

On rings of real valued clopen continuous functions

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ABSTRACT

Among variant kinds of strong continuity in the literature, the clopen continuity or *cl*-supercontinuity (i.e., inverse image of every open set is a union of clopen sets) is considered in this paper. We investigate and study the ring $C_s(X)$ of all real valued clopen continuous functions on a topological space X . It is shown that every $f \in C_s(X)$ is constant on each quasi-component in X and using this fact we show that $C_s(X) \cong C(Y)$, where Y is a zero-dimensional s -quotient space of X . Whenever X is locally connected, we observe that $C_s(X) \cong C(Y)$, where Y is a discrete space. Maximal ideals of $C_s(X)$ are characterized in terms of quasi-components in X and it turns out that X is mildly compact (every clopen cover has a finite subcover) if and only if every maximal ideal of $C_s(X)$ is fixed. It is shown that the socle of $C_s(X)$ is an essential ideal if and only if the union of all open quasi-components in X is s -dense. Finally the counterparts of some familiar spaces, such as P_s -spaces, almost P_s -spaces, s -basically and s -extremally disconnected spaces are defined and some algebraic characterizations of them are given via the ring $C_s(X)$.

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1. INTRODUCTION

If X and Y are topological spaces, then a function $f : X \rightarrow Y$ is said to be *clopen continuous* [9] or *cl-supercontinuous* [10] if for every $x \in X$ and for each open set V in Y containing $f(x)$, there exists a clopen (closed and open) set U in X containing x such that $f(U) \subseteq V$. Since this is a strong form of continuity, let us rename “clopen continuous” as *strongly continuous* and for brevity write *s-continuous*. If A is a subset of X , then s -interior of A denotes the set of all $x \in A$ such that A contains a clopen set containing x and we denote it by $\text{int}_s A$. A subset G of X is said to be s -open if $G = \text{int}_s G$. In fact a set is s -open if and only if it is a union of clopen sets. The set of all $x \in X$ such that every clopen set containing x intersects A is called s -closure of A and it is denoted by $\text{cl}_s A$. Similarly a set H is called s -closed if $H = \text{cl}_s H$ and a set is s -closed if and only if it is an intersection of clopen sets. A bijection function $\theta : X \rightarrow Y$ is called *s-homeomorphism* [10, under the name of cl-homeomorphism] if both θ and θ^{-1} are s -continuous. If such function from X onto Y exists, we say that X and Y are *s-homeomorphic* and we write $X \cong_s Y$.

We denote by $C(X)$ the ring of all real-valued continuous functions on a space X and by $C_s(X)$ the set of all real valued s -continuous functions on X . It is easy to see that $C_s(X)$ is a ring and in fact it is a subring of $C(X)$. For each $f \in C(X)$, the *zero-set* of f , denoted by $Z(f)$, is the set of zeros of f and $X \setminus Z(f)$ is the *cozero-set* of f . The set of all zero-sets in X is denoted by $Z(X)$ and we also denote by $Z_s(X)$ the set of all zero-sets $Z(f)$, where $f \in C_s(X)$. Clearly $Z_s(X) \subseteq Z(X)$. If $Z \in Z_s(X)$, then it is the inverse image of the closed subset $\{0\}$ of \mathbb{R} under an element of $C_s(X)$ and this implies that every zero-set in $Z_s(X)$ is s -closed. Hence every cozero-set $X \setminus Z(f)$, where $f \in C_s(X)$, is s -open. The converse is not necessarily true. For instance let S be an uncountable space in which all points are isolated except for a distinguish point s . Neighborhoods of s are considered to be those sets containing s with countable complement, see Problem 4N in [6]. Since $\{s\} = \bigcap_{s \neq a \in S} (S \setminus \{a\})$, the singleton $\{s\}$ is s -closed but it is not a zero-set. It is well-known that a space X is a completely regular Hausdorff space if and only if $Z(X)$ is a base for closed subsets of X , if and only if the set of all cozero-sets is a base for open subsets of X , see Theorem 3.2 in [6]. Whenever X is zero-dimensional (i.e., a T_1 space with a base consisting of clopen sets), then clearly $C(X)$ and $C_s(X)$ coincide, see also Remark 2.3 in [10]. If X is a completely regular Hausdorff space, the converse is also true, we cite this fact as a lemma for later use.

Lemma 1.1. *Whenever X is zero-dimensional, then $C(X) = C_s(X)$ and if X is a completely regular Hausdorff space, the converse is also true.*

Proof. The first implication is obvious, see also Remark 2.3 in [10]. For the converse, as we have already mentioned the collection $\mathcal{C} = \{X \setminus Z(f) : f \in C(X)\}$ is a base for open sets in X . Now let $f \in C(X)$ and $x \in X \setminus Z(f)$. Since $f(x) \neq 0$ and $f \in C(X) = C_s(X)$, there exists a clopen set U in X containing x such that $f(y) \neq 0$, for each $y \in U$. Hence $U \subseteq X \setminus Z(f)$ which means that X has a base with clopen sets. Since X is T_1 , it is zero-dimensional. \square

We recall that a completely regular Hausdorff space X is a P -space if every G_δ -set or equivalently every zero-set in X is open and it is an *almost P -space* if every non-empty G_δ -set or equivalently every nonempty zero-set in X has a non-empty interior. Hence every P -space is an almost P -space but the converse fails, for instance, the one-point compactification of an uncountable discrete space is an almost P -space which is not a P -space, see Example 2 in [8] and problem 4K(1) in [6]. Basically (extremally) disconnected spaces are those spaces in which every cozero-set (open set) has an open closure. Clearly every extremally disconnected space is basically disconnected but not conversely, see Problem 4N in [6]. Several algebraic and topological characterizations of aforementioned spaces are given in [3], [4], [6] and [8].

An ideal I in a commutative ring is called a z -ideal if $M_a \subseteq I$ for each $a \in I$, where M_a is the intersection of all maximal ideals in the ring containing a . It is easy to see that an ideal I in $C(X)$ ($C_s(X)$) is a z -ideal if and only if whenever $f \in I$, $g \in C(X)$ ($g \in C_s(X)$) and $Z(f) \subseteq Z(g)$, then $g \in I$, see Problem 4A in [6]. A non-zero ideal in a ring is said to be *essential* if it intersects every non-zero ideal non-trivially. Intersection of all essential ideals in a ring is called the *socle* of the ring. A topological characterization of essential ideals in $C(X)$ and the socle of $C(X)$ are given in [2] and [7] respectively. An ideal I in $C(X)$ or $C_s(X)$ is said to be *fixed* if $\bigcap_{f \in I} Z(f) \neq \emptyset$, otherwise it is called *free*. The space βX is the *Stone-Ćech compactification* of X and for any $p \in \beta X$, M^p (resp., O^p) is the set of all $f \in C(X)$ for which $p \in \text{cl}_{\beta X} Z(f)$ (resp., $p \in \text{int}_{\beta X} \text{cl}_{\beta X} Z(f)$).

The *component* C_x of a point x in a topological space X is the union of all connected subspaces of X which contain x . The *quasi-component* Q_x of x is the intersection of all clopen subsets of X which contain x . Clearly $C_x \subseteq Q_x$ for each $x \in X$ and the inclusion may be strict, see Example 6.1.24 in [5]. It is well-known that for any two distinct points x and y in a space X , either $Q_x = Q_y$ or $Q_x \cap Q_y = \emptyset$, see [5]. Components and quasi-components in a space X are closed and whenever X is locally connected, then they are also open, see Corollary 27.10 in [12]. The converse of this fact is not true in general, see the example which is given preceding Lemma 1.1. Whenever the components in a space X are the points, then X is called *totally disconnected*. Equivalently, X is totally disconnected if and only if the only non-empty connected subsets of X are the singleton sets.

2. SOME PROPERTIES OF CLOPEN CONTINUOUS FUNCTIONS

Behaviour of s -continuous functions on quasi-components is investigated in this section. The results of this section play an important role in the next sections.

Proposition 2.1. *Let X and Y be topological spaces and $f : X \rightarrow Y$ be a s -continuous function. Then the following statements hold.*

- (1) *If $x \in X$, $y \in Y$ and $f(x) = y$, then $f(Q_x) \subseteq Q_y$.*
- (2) *If Y is a T_1 -space, then f is constant on each quasi-component in X .*

- (3) Whenever f is one-to-one and Y is a T_1 -space, then X is totally disconnected.
- (4) If f is s -homeomorphism and $f(x) = y$, then $f(Q_x) = Q_y$. Moreover, if X or Y is T_1 , then both X and Y are totally disconnected.

Proof. (1) Let $z \in Q_x$ but $f(z) \notin Q_y$. Then there is a clopen set U in Y such that $f(z) \in U$ and $U \cap Q_y = \emptyset$ which implies $y \notin U$. Since f is s -continuous, there exists a clopen set G in X containing z such that $f(G) \subseteq U$. But $Q_x \subseteq G$ implies that $f(Q_x) \subseteq U$ and this yields that $f(x) = y \in U$, a contradiction.

(2) Let $y \in Q_x$ and $f(x) \neq f(y)$. Then there exists an open set H in Y such that $f(x) \in H$ but $f(y) \notin H$. Since f is s -continuous, there exists a clopen set G in X containing x such that $f(G) \subseteq H$. But any clopen set containing x contains y as well, for $y \in Q_x$, so $f(y) \in H$, a contradiction.

(3) Using part (2), f is constant on Q_x for every $x \in X$. But f is one-to-one, so Q_x is singleton for each $x \in X$. This implies that every quasi-component in X , and thus every component of X is singleton, i.e., X is totally disconnected.

(4) It is evident by parts (1) and (3). □

Corollary 2.2. For every $f \in C_s(X)$, the following statements hold.

- (1) f is constant on each quasi-component in X .
- (2) The zero-set $Z(f)$ is a union of quasi-components in X . In fact $Z(f) = \bigcup_{x \in Z(f)} Q_x$.

A space X is called *ultra Hausdorff* [11] if for each pair of distinct points in X , there exists a clopen set in X containing one but not the other. Two disjoint subsets of a space are called s -completely separated if there is a function $f \in C_s(X)$ which separates them. Similar to the proof of Theorem 1.15 in [6], it is easy to see that two disjoint sets are s -completely separated if and only if they are contained in two disjoint members of $Z_s(X)$. By the following proposition, ultra Hausdorff spaces are characterized by quasi-components in the spaces and every s -closed set is s -completely separated from every quasi-component disjoint from it.

Proposition 2.3. Let X be a topological space. Then the following statements hold.

- (1) If A is a s -closed subset of X and $x \in X \setminus A$, then there exists $g \in C_s(X)$ such that $g(A) = \{1\}$ and $g(Q_x) = \{0\}$.
- (2) X is ultra Hausdorff if and only if every quasi-component in X is singleton.

Proof. (1) Since $X \setminus A$ is s -open, there exists a clopen set U containing x such that $U \cap A = \emptyset$. Now define an idempotent e with $Z(e) = U$, i.e., $e(U) = \{0\}$ and $e(X \setminus U) = \{1\}$. Clearly $e \in C_s(X)$, $e(Q_x) = 0$, since $Q_x \subseteq U$ and $e(A) = 1$.

(2) Whenever X is ultra Hausdorff and $x, y \in X$ are distinct, then there exists a clopen set U containing x but not y . This implies that $y \notin Q_x$, i.e. Q_x

is singleton. Conversely, let $x, y \in X$ be distinct points. Since $Q_x = \{x\}$, there is a clopen set U such that $x \in U$ and $y \notin U$, i.e., X is ultra Hausdorff. \square

3. $C_s(X)$ IS A $C(Y)$

As an equivalent definition, a space X is zero-dimensional if and only if it is T_1 and for each point $x \in X$ and each closed subset A of X not containing x , there exists a clopen set G in X containing x such that $G \cap A = \emptyset$. Clearly every zero-dimensional space is Tychonoff. Whenever we consider the collection of all clopen subsets of (X, τ) as a base for a topology τ^* on X , then $C_s(X, \tau) = C(X, \tau^*)$, by Theorem 5.1 in [10]. But the space (X, τ^*) is not necessarily T_1 and so it may not be zero-dimensional. In the following theorem we show that $C_s(X)$ is in fact a $C(Y)$ for a zero-dimensional space Y which is also a s -quotient space of X .

Theorem 3.1. *For each topological space X , there exists a zero-dimensional space X_z such that $C_s(X) \cong C(X_z)$.*

Proof. For each $x \in X$, let Q_x be the quasi-component of x and consider the decomposition $X_z = \{Q_x : x \in X\}$. Take a topology τ on X_z so that $G \in \tau$ if and only if $\bigcup_{Q_x \in G} Q_x$ is s -open in X . To see that τ is a topology, clearly $X = \bigcup_{Q_x \in X_z} Q_x$ and $\emptyset = \bigcup_{Q_x \in \emptyset} Q_x$ imply that X_z and \emptyset are open. Whenever H and K are open sets in X_z , then $\bigcup_{Q_x \in H \cap K} Q_x = (\bigcup_{Q_x \in H} Q_x) \cap (\bigcup_{Q_x \in K} Q_x)$ imply that $H \cap K$ is open in X_z . It is also easy to see that $\bigcup_{\alpha} H_{\alpha}$ is open in X_z for each open set H_{α} in X_z . The space X_z is Hausdorff, in fact if Q_x and Q_y are two distinct points in X_z , where $x, y \in X$, then $x \notin Q_y$ and since Q_y is s -closed, there is an idempotent $e \in C_s(X)$ such that $e(Q_y) = 0$ and $e(Q_x) = 1$, by part (1) of Proposition 2.3. If we set $H = \{Q_z : z \in Z(e)\}$, then $Z(e) = \bigcup_{Q_z \in H} Q_z$ implies that H is a clopen subset of X_z . Moreover $Q_y \in H$ but $Q_x \notin H$, i.e., X_z is ultra Hausdorff and hence it is Hausdorff as well. To show that X_z is zero-dimensional, let H be a closed set in X_z and $Q_y \notin H$. Hence $T = \bigcup_{Q_x \in H} Q_x$ is a s -closed subset of X and $y \notin T$. Therefore by Proposition 2.3, there exists a clopen set U in X containing T but not containing y . Now $\bigcup_{z \in U} Q_z = U$ implies that $K = \{Q_z : z \in U\}$ is a clopen subset of X_z . Clearly $H \subseteq K$ and $Q_y \notin K$, hence X_z is zero-dimensional.

Now it remains to show that $C_s(X) \cong C(X_z)$. To this end, define $\varphi : C_s(X) \rightarrow C(X_z)$ by $\varphi(f) = f_z$ for each $f \in C_s(X)$, where f_z is defined by $f_z(Q_x) = f(x)$, for each $x \in X$. By Corollary 2.2, clearly φ is well-defined. Moreover $f_z \in C(X_z)$ for each $f \in C_s(X)$. In fact if $f_z(Q_x) = f(x) = c$, then for each $\varepsilon > 0$, there exists a clopen set G in X containing x such that $f(G) \subseteq (c - \varepsilon, c + \varepsilon)$, for $f \in C_s(X)$. Now we take the open set $H = \{Q_z : z \in G\}$ in X_z containing Q_x . Hence $f_z(H) = f(G) \subseteq (c - \varepsilon, c + \varepsilon)$ implies that $f_z \in C(X_z)$. Whenever $\varphi(f) = \varphi(g)$, $f, g \in C_s(X)$, then $f_z = g_z$ implies that $f(x) = f_z(Q_x) = g_z(Q_x) = g(x)$, for all $x \in X$. Hence $f = g$, i.e., φ is one-to-one. φ is also homomorphism, for $\varphi(f + g) = (f + g)_z$ and $(f + g)_z(Q_x) = (f + g)(x) = f(x) + g(x) = f_z(Q_x) + g_z(Q_x)$, for each $Q_x \in X_z$. This implies that

$\varphi(f+g) = \varphi(f) + \varphi(g)$, for all $f, g \in C_s(X)$ and hence φ is homomorphism. To complete the proof, we must show that φ is onto. To this end, let $g \in C(X_z)$. The function $f : X \rightarrow \mathbb{R}$ defined by $f(x) = g(Q_x)$, for all $x \in X$ is s -continuous. In fact, if $x \in X$, $f(x) = g(Q_x) = c$ and $\varepsilon > 0$ is given, then there is an open set H in X_z containing Q_x such that $g(H) \subseteq (c - \varepsilon, c + \varepsilon)$. Now it is enough to take the s -open subset $G = \bigcup_{Q_z \in H} Q_z$ of X . Clearly $x \in G$ and $f(G) \subseteq (c - \varepsilon, c + \varepsilon)$ which implies that $f \in C_s(X)$. By definitions of f and φ , it is clear that $\varphi(f) = g$ and we have thus shown that φ is onto. \square

Corollary 3.2. *If every quasi-component in X is open, in particular, if X is locally connected, then $C_s(X) \cong C(Y)$ for a discrete space Y .*

Proof. Whenever each quasi-component of X is open, then each set $\{Q_x\}$ is open in the space X_z , defined in the proof of Theorem 3.1. Therefore each Q_x is an isolated point in X_z , so $Y = X_z$ is discrete and $C_s(X) \cong C(Y)$. \square

The space X_z defined in the proof of Theorem 3.1 is a s -continuous image of X . In fact, if we regard the natural function $\tau : X \rightarrow X_z$, with $\tau(x) = Q_x$ for each $x \in X$, then τ is continuous and $f_z \circ \tau = f$ or equivalently, $\varphi(f) \circ \tau = f$. In order to prove that τ is s -continuous at $x \in X$, let H be a clopen subset of X_z containing Q_x (in fact H is an element of a base of the zero-dimensional space X_z). Now it is enough to take $U = \bigcup_{Q \in H} Q$ which is s -open in X containing x and clearly $\tau(U) = H$, i.e., τ is continuous at x . In this case we may say naturally, like the notion of quotient space, that X_z is a s -quotient of X and the induced map τ is a s -quotient map. The equality $f_z \circ \tau = f$ is evident.

We conclude this section by the following proposition.

Proposition 3.3. *For two spaces X and Y , if $X \cong_s Y$, then $X_z \cong Y_z$.*

Proof. let $\sigma : X \rightarrow Y$ be a s -homeomorphism. By Proposition 2.1, if $\sigma(x) = y$, then $\sigma(Q_x) = Q_y$. In fact every quasi-component in Y is exactly the image of a unique quasi-component in X under σ . Define $\phi : X_z \rightarrow Y_z$ by $\phi(Q) = \sigma(Q)$ for each quasi-component Q in X . Clearly ϕ is a bijection map. Given a quasi-component Q_x in X , we show that ϕ is continuous at Q_x . Let H be an open subset of Y_z containing $\phi(Q_x) = \sigma(Q_x) = Q_y$ and take $V = \bigcup_{Q \in H} Q$. Hence by definition of open sets in Y_z , V is s -open in Y containing $\sigma(Q_x) = Q_y$. Since σ is onto, there exists an element of Q_x , say x without loss of generality, such that $\sigma(x) = y$. Therefore there exists a clopen set U in X containing x (and hence containing Q_x) such that $\sigma(U) \subseteq V$. Now we set $G = \{Q_z : z \in U\}$. Clearly G is open in X_z containing Q_x and $\phi(G) \subseteq H$. Similarly, ϕ^{-1} is also continuous and we are done. \square

The converse of the Proposition 3.3 is not true. If we take the discrete space $X = \{a, b\}$ and the space $Y = (0, 1) \cup (1, 2)$ as a subspace of \mathbb{R} , then clearly $X \cong X_z$ and Y_z is a discrete space containing two elements $(0, 1)$ and $(1, 2)$. Hence $X_z \cong Y_z$, but X and Y are not s -homeomorphic.

4. MAXIMAL IDEALS OF $C_s(X)$

In this section, for each space X , we consider X_z and the isomorphism $\varphi : C_s(X) \rightarrow C(X_z)$ as defined in the proof of Theorem 3.1. First we show that φ takes fixed (free) ideals to fixed (free) ideals and using this, we transfer some well-known facts in the context of $C(X)$ to that of $C_s(X)$.

Lemma 4.1. *An ideal I in $C_s(X)$ is fixed if and only if $\varphi(I)$ is a fixed ideal in $C(X_z)$. In particular, φ takes fixed maximal ideals to fixed maximal ideals.*

Proof. If $g \in \varphi(I)$, then there exists $f \in I$ such that $g = \varphi(f) = f_z$. Now $f(x) = 0$ if and only if $g(Q_x) = f_z(Q_x) = f(x) = 0$. Therefore, $x \in \bigcap_{f \in I} Z(f)$ if and only if $Q_x \in \bigcap_{g \in \varphi(I)} Z(g) = \bigcap_{f \in I} Z(f_z)$. \square

Theorem 4.2. *For a topological space X , the fixed maximal ideals in $C_s(X)$ are precisely the sets*

$$M_{Q_x} = \{f \in C_s(X) : Q_x \subseteq Z(f)\} \quad x \in X.$$

The ideals M_{Q_x} are distinct for distinct Q_x . For each $x \in X$, $C_s(X)/M_{Q_x}$ is isomorphic with the real field \mathbb{R} .

Proof. Using Lemma 4.1, fixed maximal ideals of $C_s(X)$ are precisely of the form $\varphi^{-1}(M_y)$, where $y \in X_z$ (i.e., $y = Q_x$ for some $x \in X$). Now $f \in \varphi^{-1}(M_y)$ if and only if $f_z \in M_y = M_{Q_x}$, or equivalently $Q_x \subseteq Z(f)$, by Theorem 3.1. Hence $\varphi^{-1}(M_y) = M_{Q_x}$. Whenever $Q_p \neq Q_q$ for $p, q \in X$, using Proposition 2.3, there exists $g \in C_s(X)$ such that $g(Q_p) = 0$ and $g(Q_q) = 1$ and this means that $g \in M_{Q_p} \setminus M_{Q_q}$. Finally $\sigma : C_s(X) \rightarrow \mathbb{R}$ with $\sigma(f) = f(x)$ (note that $f(y) = f(x)$, for all $y \in Q_x$ by Corollary 2.2), for all $f \in C_s(X)$ is a homomorphism with kernel M_{Q_x} , so $C_s(X)/M_{Q_x} \cong \mathbb{R}$. \square

A space is said to be *mildly compact* [11] if every clopen cover of X has a finite subcover. Clearly every compact space is mildly compact but not conversely. For instance, consider the space $X = (0, 1) \cup (1, 2)$ as a subspace of \mathbb{R} . By the following proposition, for a space X , the compactness of X_z is equivalent to mildly compactness of X .

Proposition 4.3. *For a space X , the following statements are equivalent.*

- (1) X is mildly compact.
- (2) X_z is compact.
- (3) Every ideal of $C_s(X)$ is fixed.
- (4) Every maximal ideal of $C_s(X)$ is fixed.

Proof. A collection $\{H_\alpha : \alpha \in S\}$ is an open cover of X_z if and only if the collection $\{G_\alpha : \alpha \in S\}$, where $G_\alpha = \bigcup_{Q \in H_\alpha} Q$, is a s -open cover of X . This implies the equivalence of parts (1) and (2). The equivalence of third and fourth parts with part (1) is an immediate consequence of Lemma 4.1 and Theorem 4.11 in [6]. \square

Using Theorem 3.1, and in view of the fact that there is a correspondence between elements of the space βX_z and the set of all maximal ideals of $C(X_z)$ by Theorem 7.3 in [6], all maximal ideals of $C_s(X)$, fixed or free, will be characterize by the following theorem for each space X .

Theorem 4.4. *For every space X , the maximal ideals of $C_s(X)$ are precisely of the form*

$$M^p = \{f \in C_s(X) : p \in cl_{\beta X_z} Z(f_z)\} \quad p \in \beta X_z.$$

Remark 4.5. As in $C(X)$, for each maximal ideal M of $C_s(X)$, we define $O_M = \{f \in C_s(X) : fg = 0 \text{ for some } g \notin M\}$, see 7.12(b) in [6] and the argument preceding Theorem 2.12 in [1]. Whenever M is fixed, then $M = M_{Q_x}$ for some $x \in X$ by Theorem 4.2, and therefore we have $O_M = O_{Q_x} = \{f \in C_s(X) : Q_x \subseteq \text{int}_s Z(f)\}$. In fact, if $f \in O_M$, then $fg = 0$ for some $g \notin M_{Q_x}$. Hence $Q_x \subseteq X \setminus Z(g) \subseteq Z(f)$. But $X \setminus Z(g)$ is s -open, hence $Q_x \subseteq \text{int}_s Z(f)$, so $f \in O_{Q_x}$. Whenever $f \in O_{Q_x}$, then $Q_x \subseteq \text{int}_s Z(f)$. Since $\text{int}_s Z(f)$ is s -open, there exists a clopen set U containing Q_x contained in $\text{int}_s Z(f)$. Now if we take an idempotent e such that $Z(e) = U$, then $Q_x \subseteq Z(e) \subseteq \text{int}_s Z(f)$. Therefore $1 - e \notin M_{Q_x}$ and $(1 - e)f = 0$ which implies that $f \in O_{Q_x} = O_M$.

By Theorem 2.4 in [4], X is zero-dimensional if and only if for each $x \in X$, the ideal O_x is generated by a set of idempotents. Hence for each $y \in X_z$ ($y = Q_x$ for some $x \in X$), the ideal O_y is generated by a set of idempotents. Now in view of Theorem 2.4 in [4] and using the isomorphism φ defined in the proof of Theorem 3.1, the following corollary is evident.

Corollary 4.6. *For each $x \in X$, the ideal O_{Q_x} in $C_s(X)$ is generated by a set of idempotents.*

Remark 4.7. By Theorem 4.9 in [6], two compact spaces X and Y are homeomorphic if and only if $C(X)$ and $C(Y)$ are isomorphic. It is clear that if $X \cong_s Y$, then $C_s(X) \cong C_s(Y)$. In fact, if $\sigma : Y \rightarrow X$ is a s -homeomorphism, then $f \rightarrow f \circ \sigma$ is a s -isomorphism between $C_s(X)$ and $C_s(Y)$. But in contrast to Theorem 4.9 in [6], we observe that $C_s(X) \cong C_s(Y)$ does not necessarily imply $X \cong_s Y$ even if X and Y are mildly compact. To see this, consider spaces $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ and $Y = (\bigcup_{n=1}^{\infty} (\frac{1}{n+1}, \frac{1}{n})) \cup \{0\}$ as subspaces of \mathbb{R} . Clearly, X and Y are mildly compact. Also $C_s(X) \cong C_s(Y)$, in fact, every $g \in C_s(Y)$ is constant on each interval $(\frac{1}{n+1}, \frac{1}{n})$, by Proposition 2.1, say $g((\frac{1}{n+1}, \frac{1}{n})) = \{a_n\}$. Now if we define $f_g : X \rightarrow \mathbb{R}$ by $f_g(\frac{1}{n}) = a_n$ and $f_g(0) = g(0)$, then $f_g \in C_s(X)$ and $\theta : C_s(Y) \rightarrow C_s(X)$, $\theta(g) = f_g$ for each $g \in C_s(Y)$ is an isomorphism. Since there is no bijection function between X and Y , these two spaces are not s -homeomorphic.

Proposition 4.8. *If $X_z \cong Y_z$, then $C_s(X) \cong C_s(Y)$ and whenever X and Y are mildly compact, then the converse is also true.*

Proof. $X_z \cong Y_z$ implies that $C(X_z) \cong C(Y_z)$ and hence by Theorem 3.1, $C_s(X) \cong C_s(Y)$. For the converse, $C_s(X) \cong C_s(Y)$ implies $C(X_z) \cong C(Y_z)$

by Theorem 3.1. Now using Proposition 4.3, X_z and Y_z are compact, hence $X_z \cong Y_z$, by Theorem 4.9 in [6]. \square

5. SOME RELATIONS BETWEEN ALGEBRAIC PROPERTIES OF $C_s(X)$ AND TOPOLOGICAL PROPERTIES OF X

We call a space X a P_s -space if every zero-set in $Z_s(X)$ is open. Clearly, every P -space is a P_s -space but not conversely. For instance, $X = (0, 1) \cup (1, 2)$ as a subspace of \mathbb{R} is not a P -space whereas it is a P_s -space, for $Z_s(X) = \{\emptyset, X, (0, 1), (1, 2)\}$, by Corollary 2.2. Every P_s -space is not necessarily a completely regular space. It is enough to consider a non-completely regular space with two components.

Whenever $f \in C_s(X)$, then $Z(f_z) = \{Q_x : f(x) = 0\}$ and $Z(f) = \bigcup_{Q_x \in Z(f_z)} Q_x$. These imply, by definition of open sets in X_z , that $Z(f)$ is s -open in X if and only if $Z(f_z)$ is open in X_z . On the other hand, since $C_s(X) \cong C(X_z)$, by Theorem 3.1, the ring $C(X_z)$ is regular if and only if $C_s(X)$ is a regular ring. In view of these points, the following result is an immediate consequence of Problem 4J in [6].

Proposition 5.1. *A space X is a P_s -space if and only if $C_s(X)$ is a regular ring.*

The counterparts of the other conditions of Problem 4J in [6] can be obtained more or less for regularity of $C_s(X)$. For example, $C_s(X)$ is regular if and only if $M_{Q_x} = O_{Q_x}$, for each $x \in X$ if and only if every ideal in $C_s(X)$ is a z -ideal and so on. Note that for each $f, g \in C_s(X)$, it is easy to see that $Z(f) \subseteq Z(g)$ if and only if $Z(f_z) \subseteq Z(g_z)$ and this implies that an ideal I in $C_s(X)$ is a z -ideal if and only if $\varphi(I)$ is a z -ideal in $C(X_z)$.

We already observed that every P_s -space is not necessarily a P -space. By the following result, this happens if and only if X is zero-dimensional.

Proposition 5.2. *A space X is a P -space if and only if X is a zero-dimensional P_s -space.*

Proof. Clearly, every P -space is basically disconnected, hence using Problem 16O in [6], every P -space is zero-dimensional. Every P -space is a P_s -space as well. Conversely, since X is zero-dimensional, $C(X) = C_s(X)$ by Lemma 1.1 and since X is a P_s -space, $C_s(X)$ is a regular ring, by Proposition 5.1. This implies that $C(X)$ is also a regular ring. Now using Problem 4J in [6], X is a P -space. \square

We call a space X an almost P_s -space if every non-empty zero-set in $Z_s(X)$ has a non-empty s -interior. However the notion of almost P_s -space is the counterpart of that of almost P -space but the class of almost P -spaces and the class of almost P_s -spaces are dissimilar. The following example shows that these classes are not comparable and none of them is larger than the other.

Example 5.3. Whenever every quasi-component in a space X is open, in particular, if X is locally connected, then X is a P_s -space. In fact if $Z(f) \neq \emptyset$,

$f \in C_s(X)$, then $Z(f)$ is a union of quasi-components in X , by Corollary 2.2. Since each quasi-component in X is clopen, $Z(f)$ is s -open. This implies that $Y = (0, 1) \cup (1, 2)$ as a subspace of \mathbb{R} is a P_s -space. Hence Y is an almost P_s -space but clearly, it is not an almost P -space. Also every almost P -space need not be an almost P_s -space. To see this let X be a (completely regular Hausdorff) connected almost P -space, see Proposition 2.3 in [8] for existence of such a space. Take a point $\sigma \in X$ and let $Y = X \cup \mathbb{N}$ with the topology as follows: elements of \mathbb{N} are considered to be isolated points, neighborhoods of all points of X , except σ will be the same as in the space X while each neighborhood of σ in Y will be of the form $G \cup A$, where G is an open set in X containing σ and A is a subset of \mathbb{N} such that $\mathbb{N} \setminus A$ is finite. If we define $f : Y \rightarrow \mathbb{R}$ with $f(n) = \frac{1}{n}$ and $f(X) = \{0\}$, then $f \in C_s(Y)$. Since X is connected and $X = \bigcap_{n=1}^{\infty} (Y \setminus \{n\})$, it is a quasi-component in Y . Also it does not contain any clopen subset of Y (note that X itself is not open in Y , for σ is the cluster point of the subset \mathbb{N} of Y and hence \mathbb{N} is not closed in Y). Hence $\text{int}_s Z(f) = \emptyset$, i.e., Y is not an almost P_s -space.

It remains to show that Y is an almost P -space. Let $f \in C(Y)$. If $Z(f) \cap \mathbb{N} \neq \emptyset$, then clearly $\text{int}_Y Z(f) \neq \emptyset$. Now suppose that $Z(f) \cap \mathbb{N} = \emptyset$. Whenever $\sigma \notin Z(f)$, then $\text{int}_Y Z(f) = \text{int}_X Z(f|_X) \neq \emptyset$, for X is an almost P -space. Finally, suppose that $\sigma \in Z(f)$, then $Z(f) \neq \{\sigma\}$, for otherwise $\text{int}_X Z(f|_X) = \text{int}_X \{\sigma\} \neq \emptyset$ implies that σ is an isolated point of X which contradicts the connectedness of X . Therefore there exists $x \neq \sigma$ such that $x \in Z(f)$. Since Y is completely regular Hausdorff, define $h \in C(Y)$ so that $h(x) = 0$ and $h(\sigma) = 1$. Now take $g = f^2 + h^2$, then $\sigma \notin Z(g) \subseteq X$ implies that $\emptyset \neq \text{int}_X Z(g) = \text{int}_Y Z(g) \subseteq \text{int}_Y Z(f)$. Hence Y is an almost P -space.

For the proof of the following proposition, we need the following lemma.

Lemma 5.4. *Let $f, g \in C_s(X)$.*

- (1) *If $Z(g) \subseteq \text{int}_s Z(f)$, then f is a multiple of g .*
- (2) *$Z_s(X)$ is closed under countable intersection.*

Proof. (1) Let X_z, φ and f_z for each $f \in C_s(X)$ be as in the proof of Theorem 3.1. Let $Q_x \in Z(g_z)$, where $x \in X$. Hence $x \in Z(g)$ and so $x \in \text{int}_s Z(f)$, by our hypothesis. Since $\text{int}_s Z(f)$ is s -open, there exists a clopen set U such that $x \in U \subseteq Z(f)$. Now $H = \{Q_y : y \in U\}$ is clopen in X_z and $Q_x \in H \subseteq Z(f_z)$. This implies that $Z(g_z) \subseteq \text{int}_{X_z} Z(f_z)$ and using Problem 1D in [6], there is $k_z \in C(X_z)$, where $k \in C_s(X)$ such that $f_z = k_z g_z$. Now it is clear that $f = kg$.

(2) It is easy to see that whenever $\{S_n\}$ is a sequence in $C_s(X)$ converges uniformly to a function f , then $f \in C_s(X)$. Now, as in 1.14(a) in [6], if for each $n \in \mathbb{N}$, we consider $Z_n = Z(f_n)$, where $f_n \in C_s(X)$ and $|f_n| \leq 1$ (note that if $f \in C_s(X)$, then $\frac{f}{1+|f|} \in C_s(X)$, $Z(\frac{f}{1+|f|}) = Z(f)$ and $|\frac{f}{1+|f|}| \leq 1$), then the sequence $S_n = \sum_{i=1}^n f_i/2^i$ converges uniformly to a function $f \in C_s(X)$. Clearly $\bigcap_{n=1}^{\infty} Z(f_n) = Z(f)$. \square

Proposition 5.5. *For a topological space X , the following statements are equivalent.*

- (1) X is an almost P_s -space.
- (2) For each non-unit $f \in C_s(X)$, $f = ef$ for some idempotent $e \neq 1$ in $C_s(X)$.
- (3) For each non-unit $f \in C_s(X)$, $fe = 0$ for some idempotent $e \neq 0$ in $C_s(X)$.
- (4) Every non-empty countable intersection of s -open sets has a non-empty s -interior.

Proof. (1) \Rightarrow (2) If $f \in C_s(X)$ is not unit, then $\text{int}_s Z(f) \neq \emptyset$. Since $\text{int}_s Z(f)$ is s -open, there exists a non-empty clopen set U contained in $Z(f)$. Take the idempotent e with $Z(e) = U$. Clearly $e \neq 1$, for $U \neq \emptyset$. Since $Z(e) \subseteq \text{int}_s Z(f)$, f is a multiple of e , by Lemma 5.4. Hence $f = eg$ for some $g \in C_s(X)$. But $f = g$ on $X \setminus Z(e)$, so $f = ef$.

(2) \Rightarrow (3) $f = ef$ implies $f(1 - e) = 0$, where $1 - e$ is a non-zero idempotent.

(3) \Rightarrow (4) Let $A = \bigcap_{n=1}^{\infty} A_n \neq \emptyset$, where each A_n is s -open. Let $x \in A$. Hence there is an idempotent $e_n \in C_s(X)$ such that $x \in Z(e_n) \subseteq A_n$, for each $n \in \mathbb{N}$. Now by Lemma 5.4, $\bigcap_{n=1}^{\infty} Z(e_n)$ is a zero-set, say $Z(g)$, where $g \in C_s(X)$. Since g is non-unit ($x \in Z(g)$), there exists an idempotent $0 \neq e \in C_s(X)$ such that $ge = 0$, by our hypothesis. Therefore $\emptyset \neq Z(1 - e) \subseteq Z(g) \subseteq A$ which means that A has a non-empty s -interior.

(4) \Rightarrow (1) Since every zero-set in $Z_s(X)$ is a countable intersection of s -open sets, the proof is evident. We note that whenever $f \in C_s(X)$, then $Z(f) = \bigcap_{n=1}^{\infty} f^{-1}((-\frac{1}{n}, \frac{1}{n}))$ and each $f^{-1}((-\frac{1}{n}, \frac{1}{n}))$ is s -open, by Theorem 2.2 in [10]. \square

We call a space X s -basically (s -extremally) disconnected if for every zero-set $Z(f) \in Z_s(X)$ (s -closed subset H of X), $\text{int}_s Z(f)$ ($\text{int}_s H$) is s -closed. Equivalently, X is a s -basically (s -extremally) disconnected space if and only if for each $X \setminus Z(f)$, $f \in C_s(X)$ (s -open subset G of X), $\text{cl}_s(X \setminus Z(f))$ ($\text{cl}_s G$) is s -open. We show the counterparts of Theorems 3.3 and 3.5 in [4] that the s -basically (s -extremally) disconnectedness of X is equivalent to saying that $C_s(X)$ is a $p.p.$ ring (Baer ring). Recall that a ring R is said to be $p.p.$ ring (Baer ring) if for each $a \in R$ ($S \subseteq R$), $\text{Ann}(a)$ ($\text{Ann}S$) is generated by an idempotent, where $\text{Ann}(a) := \{r \in R : ar = 0\}$ ($\text{Ann}S := \{r \in R : rs = 0, \forall s \in S\}$). First we need the following lemma.

Lemma 5.6. *Let X be a topological space and X_z be the space mentioned in the proof of Theorem 3.1.*

- (1) If $f \in C_s(X)$, then $\bigcap_{g \in \text{Ann}(f)} Z(g) = \text{cl}_s(X \setminus Z(f))$.
- (2) X is s -extremally (s -basically) disconnected if and only if X_z is extremally (basically) disconnected.

Proof. (1) Since $\bigcap_{g \in \text{Ann}(f)} Z(g) = \bigcap_{X \setminus Z(f) \subseteq Z(g)} Z(g)$, we have

$$X \setminus Z(f) \subseteq \bigcap_{g \in \text{Ann}(f)} Z(g).$$

But $\bigcap_{g \in \text{Ann}(f)} Z(g)$ is s -closed, hence $\text{cl}_s(X \setminus Z(f)) \subseteq \bigcap_{g \in \text{Ann}(f)} Z(g)$. Conversely, let $x \in \bigcap_{g \in \text{Ann}(f)} Z(g)$ but $x \notin \text{cl}_s(X \setminus Z(f))$. Hence there exists an idempotent $e \in C_s(X)$ such that $e(x) = 1$ and $e(\text{cl}_s(X \setminus Z(f))) = 0$, by Proposition 2.3. This implies that $e \in \text{Ann}(f)$, but $e(x) = 1$ yields that $x \notin \bigcap_{g \in \text{Ann}(f)} Z(g)$, a contradiction.

(2) Let X be s -extremally disconnected and H be an open set in X_z . Then $G = \bigcup_{Q_x \in H} Q_x = \{x \in X : Q_x \in H\}$ is a s -open set in X . Hence $\text{cl}_s G$ is s -open, by our hypothesis. Now let $Q_x \in \text{cl}_{X_z} H$. Then every clopen set in X_z containing Q_x intersects H and this implies that every clopen set in X containing x intersects G as well. Therefore $x \in \text{cl}_s G$. But $\text{cl}_s G$ is s -open, then there exists a clopen set U in X containing x contained in $\text{cl}_s G$. Now $V = \{Q_y : y \in U\}$ is a clopen set in X_z containing Q_x contained in $\text{cl}_{X_z} H$, i.e., $\text{cl}_{X_z} H$ is open in X_z and hence X_z is extremally disconnected. The proof of the converse is similar. In case of s -basically disconnectedness, the proof goes along the lines of the above arguments, so it is left to the reader. \square

Proposition 5.7. *Let X be a topological space.*

- (1) $C_s(X)$ is a p.p. ring if and only if X is a s -basically disconnected space.
- (2) $C_s(X)$ is a Baer ring if and only if X is a s -extremally disconnected space.

Proof. (1) We may apply each part of Lemma 5.6, we prefer to use part (1). If $C_s(X)$ is a p.p. ring, then for each $f \in C_s(X)$, $\text{Ann}(f) = (e)$ for some idempotent e . Now by Lemma 5.6, $Z(e) = \bigcap_{g \in \text{Ann}(f)} Z(g) = \text{cl}_s(X \setminus Z(f))$ which implies that $\text{cl}_s(X \setminus Z(f))$ is clopen and hence it is s -open. Therefore X is a s -basically disconnected space.

Conversely, suppose that X is s -basically disconnected. Hence $\bigcap_{g \in \text{Ann}(f)} Z(g) = \text{cl}_s(X \setminus Z(f))$ is s -open and hence it is clopen. Now take an idempotent e with $Z(e) = \text{cl}_s(X \setminus Z(f))$. Since $X \setminus Z(f) \subseteq \text{cl}_s(X \setminus Z(f)) = Z(e)$, we have $ef = 0$, i.e., $e \in \text{Ann}(f)$. On the other hand if $g \in \text{Ann}(f)$, then $Z(e) = \text{cl}_s(X \setminus Z(f)) \subseteq Z(g)$ implies that $Z(e) \subseteq \text{int}_s Z(g)$ and by Lemma 5.4, $g \in (e)$, i.e., $\text{Ann}(f) \subseteq (e)$.

(2) If $C_s(X)$ is a Baer ring, then $C(X_z)$ is also a Baer ring, for $C_s(X) \cong C(X_z)$, by Theorem 3.1. Now by Theorem 2.5 in [4], X_z is extremally disconnected. Thus using Lemma 5.6, X is s -extremally disconnected. The proof of the converse is similar. \square

It is manifest that every basically (extremally) disconnected space is a s -basically (s -extremally) disconnected space. The converse is not true in general. For example let $X = (0, 1) \cup (1, 2)$ be as a subspace of \mathbb{R} . In fact X is a P_s -space which is not basically disconnected and it is not extremally disconnected

as well. It is not hard to see that every s -basically disconnected almost P_s -space is a P_s -space. The following result states that the zero-dimensionality and s -basically (s -extremally) disconnectedness is equivalent to basically (extremally) disconnectedness.

Proposition 5.8. *A space is basically (extremally) disconnected if and only if it is s -basically (s -extremally) disconnected zero-dimensional.*

Proof. By Problem 16O in [6], every basically (extremally) disconnected space is zero-dimensional and since every basically (extremally) disconnected space is also s -basically (s -extremally) disconnected, the left-to-right implication is immediate. For the converse, whenever X is zero-dimensional, then by Lemma 1.1, $C(X) = C_s(X)$. Now if X is s -basically (s -extremally) disconnected, then by Proposition 5.7, $C(X) = C_s(X)$ is a *p.p.* (Baer) ring. Now by Theorems 3.3 and 3.5 in [4], X is basically (extremally) disconnected. \square

The socle $C_F(X)$ of $C(X)$ which is the intersection of all essential ideals in $C(X)$ is the set of all functions which vanish everywhere except on a finite number of isolated points of X , see Proposition 3.3 in [7]. Corollary 2.3 in [2] and Proposition 2.1 in [7] show that the socle of $C(X)$ is essential if and only if the set of isolated points of X is dense in X .

Proposition 5.9. *Let X be a topological space and X_z be the space defined in the proof of Theorem 3.1.*

- (1) *The socle $S_s(X)$ of $C_s(X)$ is free if and only if every quasi-component in X is open, if and only if X_z is discrete.*
- (2) *The socle of $C_s(X)$ is essential if and only if the union of open quasi-components in X is s -dense in X (A subset D of X is called s -dense in X if every non-empty s -open subset of X intersects D).*

Proof. We remind the reader that a subset H of X_z is open if and only if $\bigcup_{Q_x \in H} Q_x$ is s -open in X . This implies that for each $x \in X$, the quasi-component Q_x is an isolated point of X_z if and only if Q_x is s -open in X and hence it should be clopen. Since $C_F(X_z)$ is the set of all functions in $C(X_z)$ which vanish everywhere except on a finite set of isolated points of X_z , $S_s(X)$ will be the set of all functions in $C_s(X)$ which vanish everywhere except on a finite union of open quasi-components in X . Therefore, $\bigcap_{f \in S_s(X)} Z(f)$ is the union of all non-open quasi-components in X . Now it is clear that $S_s(X)$ is free if and only if every quasi-components in X is open and this is equivalent to saying that X_z is discrete. For the proof of part (2), it is easy to see that the density of isolated points of X_z is equivalent to the density of the union of open quasi-components in X . Since $C_s(X) \cong C(X_z)$, the socle of $C(X_z)$ is essential if and only if the socle of $C_s(X)$ is. Now using Corollary 2.3 in [2], we are done. \square

It is known that every extremally disconnected P -space of non-measurable cardinal is discrete, see Problem 12H in [6]. We conclude the paper by the counterpart of this fact.

Proposition 5.10. *Every quasi-component in a s -extremally disconnected P_s -space of non-measurable cardinal is open.*

Proof. Let X be s -extremally disconnected P_s -space of non-measurable cardinal. Then $C_s(X)$ is a Baer ring by Proposition 5.7. But using Proposition 3.1, $C_s(X) \cong C(X_z)$, hence $C(X_z)$ is also a Baer ring. Therefore X_z is extremally disconnected, by Theorem 3.5 in [4]. On the other hand, since X is P_s -space, $C_s(X)$ will be regular by Proposition 5.1, whence $C(X_z)$ is also regular and hence X_z is a P -space by Problem 4J in [6]. Finally, $|X_z| \leq |X|$ implies that the cardinal of X_z is non-measurable, see part (i) in the proof of Theorem 12.5 in [6]. Now X_z is extremally disconnected P -space of non-measurable cardinal which means that X_z is discrete, by Problem 12H in [6]. Therefore each Q_x is an isolated point in X_z and hence each Q_x should be open in X . \square

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