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$C_\infty(X)$ AND RELATED IDEALS

Abstract

We have characterized the spaces X for which the smallest z-ideal containing $C_\infty(X)$ is prime. It turns out that $C_\infty(X)$ is a z-ideal in $C(X)$ if and only if every zero-set contained in an open locally compact σ-compact set is compact. Some interesting ideals related to $C_\infty(X)$ are introduced and corresponding to the relations between these ideals and $C_\infty(X)$, topological spaces X are characterized. Some compactness concepts are explicitly stated in terms of ideals related to $C_\infty(X)$. Finally we have shown that a σ-compact space X is Baire if and only if every ideal containing $C_\infty(X)$ is essential.

1 Introduction.

In this article we denote by $C(X)$ ($C^*(X)$) the ring of all (bounded) real valued continuous functions on a completely regular Hausdorff space X. For every $f \in C(X)$, the zero-set $Z(f)$ is the zeros of f and an ideal I in $C(X)$ is said to be a z-ideal if $Z(f) = Z(g)$, where $f \in C(X)$ and $g \in I$, implies that $f \in I$. An ideal I in $C(X)$ is called free if $\bigcap Z[I] = \bigcap_{f\in I} Z(f) = \emptyset$, otherwise fixed. Fixed maximal ideals of $C(X)$ are the sets $M_p = \{f \in C(X) : f(p) = 0\}$, for $p \in X$. More generally, the maximal ideals of $C(X)$ free or fixed, are the sets $M^p = \{f \in C(X) : p \in \mathrm{cl}_{\beta X} Z(f)\}$, where $p \in \beta X$ and βX is the Stone-Čech compactification of X. The maximal ideals of $C^*(X)$ are precisely the sets $M^{*p} = \{f \in C^*(X) : f^\beta(p) = 0\}$, where $p \in \beta X$ and f^β is the extension of f to βX, see [8] for more details. The intersection of all free maximal ideals in $C^*(X)$, i.e., $\bigcap_{p\in\beta X\setminus X} M^{*p}$ is denoted by $C_\infty(X)$ which

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precisely consists of all continuous functions f in $C(X)$ vanishing at infinity, i.e., $\{x \in X : |f(x)| \geq \frac{1}{n}\}$ is compact, for all $n \in \mathbb{N}$, see [8]. $C_\infty(X)$ is investigated as a ring in [2] and as an ideal of $C(X)$ in [5]. If we denote $C_R(X) = \bigcap_{p \in \nu X \setminus X} M^p$, where νX is the realcompactification of X, then clearly $C_R(X)$ is a z-ideal and $C_\infty(X) = \bigcap_{p \in \beta X \setminus X} M^{*p} \subseteq \bigcap_{p \in \nu X \setminus X} M^{*p} = \bigcap_{p \in \nu X \setminus X} M^p \cap C^*(X) = C_R(X) \cap C^*(X) \subseteq C_R(X)$. (note that $M^p \cap C^*(X) = M^{*p}$ if and only if $p \in \nu X$, see 7.9 in [8]). In [2], it is shown that for a locally compact space X, $C_\infty(X) = C_R(X)$ if and only if X is a pseudocompact space. The smallest z-ideal containing $C_\infty(X)$ is the ideal $C_{l\sigma}(X) = \{f \in C(X) : X \setminus Z(f)$ is locally compact σ-compact$\}$, see [2]. The set $C_K(X)$ of all functions in $C(X)$ with compact support is the intersection of all free ideals in $C(X)$ and of all free ideals in $C^*(X)$, see [8]. So $C_K(X) \subseteq C_\infty(X) \subseteq C_{l\sigma}(X) \subseteq C_R(X)$. Topological spaces X for which $C_K(X)$ and $C_\infty(X)$ and also $C_R(X)$ and $C_\infty(X)$ coincide, are characterized in [5] and [2] respectively. In this article we characterize topological spaces X for which $C_{l\sigma}(X) = C_\infty(X)$. In [11], Mandelker has shown that $C_\psi(X)$ consisting of all functions with pseudocompact support is an ideal in $C(X)$. It is easy to see that $C_K(X) \subseteq C_\psi(X)$. Whenever $C_K(X) = C_\psi(X)$, then the space X is called ψ-compact, see [11] and [9] for more details. In [5], it is shown that $C_\infty(X) \subseteq C_\psi(X)$ if and only if $C_\infty(X)$ is an ideal of $C(X)$ and for a locally compact Hausdorff space X, $C_\infty(X) = C_\psi(X)$ if and only if X is compact. Another ideal related to $C_\infty(X)$ is the intersection of all free maximal ideals of $C(X)$ which we denote by $I(X)$, see also [11]. For any space X, we have $C_K(X) \subseteq I(X) \subseteq C_\psi(X)$. When $C_K(X) = I(X)$ or $I(X) = C_\psi(X)$ it is said that X is μ-compact or η-compact respectively. In Theorem 3.2 in [11] it is shown that $I(X) = C_\psi(X) \cap C_\infty(X)$. We show that $C_\infty(X) = C_\psi(X)$ if and only if X is η-compact and every open locally compact subset of X is relatively pseudocompact. We will introduce some other interesting ideals in $C(X)$ and $C^*(X)$ related to $C_\infty(X)$ and we give some topological characterizations corresponding to the relations between these ideals and $C_\infty(X)$.

We need the following lemma which is proved in [5].

Lemma 1.1. *Let A be an open subset of X. Then $A = X \setminus Z(f)$ for some $f \in C_\infty(X)$ if and only if A is a locally compact σ-compact subset of X.*

By X we always mean a completely regular Hausdorff space, and the reader is referred to [8] and [12] for undefined terms and notations.

2 Ideals related to $C_\infty(X)$.

Lemma 2.1. *For any space X consider the following sets:*

(a). $C_l(X) = \{f \in C(X) : X \setminus Z(f)$ *is locally compact*$\}$.

(b). $C_{\bar{l}}(X) = \{f \in C(X) : \mathrm{cl}(X \setminus Z(f))$ *is locally compact*$\}$.

(c). $C_\sigma(X) = \{f \in C(X) : X \setminus Z(f)$ *is* σ*-compact*$\}$.

(d). $C_{\bar{\sigma}}(X) = \{f \in C(X) : \mathrm{cl}(X \setminus Z(f))$ *is* σ*-compact*$\}$.

(e). $I_{\overline{l\sigma}}(X) = \{f \in C(X) : \mathrm{cl}\,(X \setminus Z(f))$ *is contained in an open locally compact* σ*-compact set*$\}$.

(f). $C_{\overline{l\sigma}}(X) = \{f \in C(X) : \mathrm{cl}(X \setminus Z(f))$, *is locally compact* σ*-compact*$\}$.

(g). $C^*_{l\sigma}(X) = \{f \in C^*(X) : X \setminus Z(f)$ *is locally compact* σ*-compact*$\}$.

Then $C^*_{l\sigma}(X)$ *is an ideal of* $C^*(X)$ *and the others are z-ideals in* $C(X)$.

PROOF. We note that the union of two open (or closed) locally compact subsets of X is locally compact. Moreover, if $X \setminus Z(f) \subseteq A$ and A is σ-compact, then clearly $X \setminus Z(f)$ is also σ-compact for it is an F_σ-set. Now $X \setminus Z(f - g) \subseteq (X \setminus Z(f)) \cup (X \setminus Z(g))$ and $X \setminus Z(fg) \subseteq X \setminus Z(f)$ imply that $C_l(X)$ and $C_{\bar{l}}(X)$ are ideals in $C(X)$. On the other hand, since every closed subset of a σ-compact set is a σ-compact, $C_\sigma(X)$, $C_{\bar{\sigma}}(X)$, $I_{\overline{l\sigma}}(X)$, $C_{\overline{l\sigma}}(X)$ and $C^*_{l\sigma}(X)$ are also ideals. It is clear that, these ideals are z-ideals. □

Lemma 2.2.

1. $I_{\overline{l\sigma}}(X) \subseteq C_{\overline{l\sigma}}(X) \subseteq C_{l\sigma}(X) \subseteq C_l(X)$.
2. $I_{\overline{l\sigma}}(X) \subseteq C_\infty(X)C(X) \subseteq C_{l\sigma}(X)$.
3. $C_K(X) = C_{\bar{\sigma}}(X) \cap C_\psi(X)$.
4. $C_{l\sigma}(X) = C_l(X) \cap C_\sigma(X) \subseteq C_l(X) \cap C_R(X)$.
5. $C_K(X) \subseteq C_{\bar{l}}(X) \subseteq C_l(X)$.

PROOF. If $f \in I_{\overline{l\sigma}}(X)$, then $\mathrm{cl}_X(X \setminus Z(f)) \subseteq A$, where A is an open locally compact σ-compact set. Then $A = X \setminus Z(g)$, for some $g \in C_\infty(X)$, by Lemma 1.1 and hence $Z(g) \subseteq \mathrm{int}_X Z(f)$ implies that f is a multiple of g, i.e., $f \in C_\infty(X)C(X)$. The proof of other inclusions of parts 1 and 2 are easy. To prove part (3), let $f \in C_{\bar{\sigma}}(X) \cap C_\psi(X)$, then $\mathrm{cl}_X(X \setminus Z(f))$ is σ-compact pseudocompact which is compact. $C_K(X) \subseteq C_{\bar{\sigma}}(X) \cap C_\psi(X)$ and part 4 and 5 are obvious. □

In part (2), whenever X is locally compact σ-compact, then we have $C_\infty(X)C(X) = C_{l\sigma}(X) = C(X)$. If X is neither locally compact nor σ-compact, the equality $C_\infty(X)C(X) = C_{l\sigma}(X)$ may also happens. For example let $X = (0,1) \cup Y$, where $Y = \{r \in \mathbb{R} : r > 1 \text{ is irrational}\}$. If $f \in C_{l\sigma}(X)$, since $X \setminus Z(f)$ is an open locally compact subset of X, $X \setminus Z(f) \subseteq L = (0,1)$. Now consider $g \in C(X)$, such that $g((0,1)) = \{1\}$ and $g(Y) = \{0\}$. Since $X \setminus Z(g)$ is locally compact σ-compact, by Lemma 1.1, $X \setminus Z(g) = X \setminus Z(h)$, for some $h \in C_\infty(X)$. Therefore $Z(g) = Z(h)$ and g is a multiple of h, for $Z(g) = Z(h)$ is open. Thus, for every $f \in C_{l\sigma}(X)$, we have $Z(h) = Z(g) \subseteq Z(f)$ which implies that f is a multiple of h, i.e., $f \in C_\infty(X)C(X)$ and hence $C_\infty(X)C(X) = C_{l\sigma}(X)$.

Proposition 2.3.

1. *$I(X) = C_{\overline{\sigma}}(X)$ if and only if X is μ-compact.*
2. *$C_\psi(X) \subseteq C_\infty(X)$ if and only if X is η-compact. Hence $C_\psi(X) = C_\infty(X)$ if and only if X is η-compact and every open locally compact set is relatively pseuodocompact.*
3. *$C_\psi(X) \subseteq C_{\overline{\sigma}}(X)$ if and only if X is ψ-compact.*

Proof. 1. $I(X) = C_\infty(X) \cap C_\psi(X) = C_{\overline{\sigma}}(X)$ if and only if $C_\infty(X) \cap C_\psi(X) = C_{\overline{\sigma}}(X) \cap C_\psi(X) = C_K(X)$ if and only if $I(X) = C_K(X)$ which means that X is μ-compact.
2. $C_\psi(X) \subseteq C_\infty(X)$ implies that $I(X) = C_\infty(X) \cap C_\psi(X) \supseteq C_\psi(X)$, i.e, X is η-compact. Conversely, if X is η-compact, then $C_\infty(X) \cap C_\psi(X) = I(X) = C_\psi(X)$ implies that $C_\psi(X) \subseteq C_\infty(X)$. Second part of (2) is obvious by Theorem 1.3 and Proposition 2.4 in [5].
3. It follows by part (3) of Lemma 2.2. □

In the following theorem we characterize spaces X for which the smallest z-ideal containing $C_\infty(X)$ is a prime ideal. We call a point $x \in X$ an l-point if x has a compact neighborhood, clearly the set of all l-points of X is open.

Theorem 2.4. *$C_{l\sigma}(X)$ is a prime ideal if and only if X has at most one non-l-point $x^* \in X$ and for any two disjoint cozerosets, one which does not contain the non-l-point, is locally compact σ-compact.*

Proof. Let $C_{l\sigma}(X)$ be a prime ideal and x^*, y^* be two different points in X with no compact neighborhood. Suppose U and V are two disjoint open sets containing x^* and y^* respectively. Define $f, g \in C(X)$ such that $f(x^*) = 1$, $f(X \setminus U) = \{0\}$ and $g(y^*) = 1$, $g(X \setminus V) = \{0\}$. Then $X \setminus Z(f) \subseteq U$, $X \setminus Z(g) \subseteq V$ and hence these two cozerosets are not locally compact, i.e.,

$f \notin C_{l\sigma}(X)$, $g \notin C_{l\sigma}(X)$, but $fg = 0 \in C_{l\sigma}(X)$. This shows that $C_{l\sigma}(X)$ is not prime, a contradiction. Thus there exists at most one $x^* \in X$ which has no compact neighborhood. Now let $(X \setminus Z(f)) \cap (X \setminus Z(g)) = \emptyset$. Hence $fg = 0$ implies that $f \in C_{l\sigma}(X)$ or $g \in C_{l\sigma}(X)$, i.e., either $X \setminus Z(f)$ or $X \setminus Z(g)$ is locally compact σ-compact. Clearly x^* does not belong to that one which is locally compact σ-compact. Conversely, let $fg = 0$. Hence $(X \setminus Z(f)) \cap (X \setminus Z(g)) = \emptyset$, and consequently one of these cozerosets does not contain any non-l-point, say $X \setminus Z(f)$. Therefore $X \setminus Z(f)$ is locally compact σ-compact, i.e., $f \in C_{l\sigma}(X)$. Since $C_{l\sigma}(X)$ is a z-ideal, then it is a prime ideal, by Theorem 2.9 in [8]. □

Example 2.5. Let S be an uncountable space in which all points are isolated points except for a distinguished point s^*, a neighborhood of s^* being any set containing s^* whose complement is countable. The only point of S with no compact neighborhood is s^* and if $(X \setminus Z(f)) \cap (X \setminus Z(g)) = \emptyset$, then s^* is not contained in one of these two cozerosets, say $X \setminus Z(g)$. Thus $g(s^*) = 0$ and since $Z(g)$ is a G_δ-set, then $X \setminus Z(g)$ is countable and hence it is σ-compact. Now by Theorem 2.4, $C_{l\sigma}(S)$ is a prime ideal.

Proposition 2.6. $C^*_{l\sigma}(X) = C_\infty(X)$ *if and only if every zero-set contained in an open locally compact σ-compact subset of X is compact.*

PROOF. Let G be an open locally compact σ-compact subset of X, and $Z = Z(g) \subseteq G$, for some $g \in C(X)$. By Lemma 1.2, there exists $f \in C_\infty(X)$ such that $X \setminus Z(f) = G$. Hence $Z(f)$ and $Z(g)$ are completely separated, and therefore there exists $h \in C^*(X)$ such that $h(Z(g)) = 1$ and $h(Z(f)) = 0$. Now $Z(f) \subseteq Z(h)$ implies that $Z(fh) = Z(h)$. Since $fh \in C_\infty(X) \subseteq C^*_{l\sigma}(X)$, $X \setminus Z(fh)$ is locally compact σ-compact and consequently $X \setminus Z(h)$ is locally compact σ-compact. Therefore $h \in C^*_{l\sigma}(X) = C_\infty(X)$. Since $Z(g) \subseteq \{x \in X : |h(x)| \geq 1\}$ and $\{x \in X : |h(x)| \geq 1\}$ is compact, $Z(g)$ is also compact. Conversely, suppose that every zero-set contained in an open locally compact σ-compact subset of X is compact and let $f \in C^*_{l\sigma}(X)$. Then $\{x \in X : |f(x)| \geq \frac{1}{n}\} \subseteq X \setminus Z(f)$. Now $X \setminus Z(f)$ is locally compact σ-compact and $\{x \in X : |f(x)| \geq \frac{1}{n}\}$ is a zero-set. This implies that $\{x \in X : |f(x)| \geq \frac{1}{n}\}$ is compact, i.e., $f \in C_\infty(X)$. Hence $C_\infty(X) = C^*_{l\sigma}(X)$. □

By a similar proof, we have the following result.

Corollary 2.7. $C_\infty(X) = C_{l\sigma}(X)$, *i.e.,* $C_\infty(X)$ *is a z-ideal in* $C(X)$, *if and only if every zero-set contained in an open locally compact σ-compact subset of X is compact.*

The following theorem shows that for some spaces such as $X = \mathbb{Q} \cup [0,1]$, we have $I(X) = C_{l\sigma}(X)$.

Theorem 2.8. *$I(X) = C_{l\sigma}(X)$ if and only if for every open locally compact σ-compact subset A of X, $\mathrm{cl}_X A$ is pseudocompact and every zero-set in A is compact.*

Proof. Let $I(X) = C_{l\sigma}(X)$. Hence $C_{l\sigma}(X) = I(X) \subseteq C_\infty(X) \cap C_\psi(X) \subseteq C_{l\sigma}(X) \cap C_\psi(X)$. Therefore $C_{l\sigma}(X) \subseteq C_\psi(X)$, i.e., every open locally compact σ-compact subset of X has a pseudocompact closure. On the other hand $I(X) = C_{l\sigma}(X)$ implies that $C_{l\sigma}(X) = C_\infty(X)$, i.e., every zero-set contained in an open locally compact σ-compact subset of X is compact. Conversely the first condition implies that $C_{l\sigma}(X) \subseteq C_\psi(X)$. Now by the second condition we have $C_\infty(X) = C_{l\sigma}(X)$. Hence $I(X) = C_{l\sigma}(X)$. □

Corollary 2.9. *Let X be a realcompact space. Then every open locally compact σ-compact subset of X has compact closure if and only if $I(X) = C_{l\sigma}(X)$.*

Proof. If X is realcompact, then $C_K(X) = I(X)$, see Theorem 8.19 in [8]. □

More generally, since $I(X) = \cap_{p \in \beta X \setminus X} M^p = C_\psi(X) \cap C_\infty(X)$, we have the following result.

Proposition 2.10. *A locally compact σ-compact open set G in X has pseudocompact closure if and only if $\beta X \setminus X \subseteq \mathrm{cl}_{\beta X}(X \setminus G)$. In particular, $\beta X \setminus X \subseteq \mathrm{cl}_{\beta X} Z(f)$ if and only if $X \setminus Z(f)$ is locally compact σ-compact and $\mathrm{cl}_{\beta X}(X \setminus Z(f))$ is pseudocompact.*

Proof. If G is locally compact σ-compact with pseudocompact closure, then $G = X \setminus Z(f)$ for some $f \in C_\infty(X)$, by Lemma 1.1. Moreover, $f \in C_\psi(X)$ for $\mathrm{cl}_X(X \setminus Z(f))$ is pseudocompact. Hence $f \in C_\infty(X) \cap C_\psi(X) = I(X) = \bigcap_{p \in \beta X \setminus X} M^p$, i.e., $\beta X \setminus X \subseteq \mathrm{cl}_{\beta X} Z(f) = \mathrm{cl}_{\beta X}(X \setminus G)$. Conversely, if G is locally compact σ-compact and $\beta X \setminus X \subseteq \mathrm{cl}_{\beta X}(X \setminus G)$, then $G = X \setminus Z(f)$ for some $f \in C_\infty(X)$ by Lemma 1.1 and hence $\beta X \setminus X \subseteq \mathrm{cl}_{\beta X} Z(f)$ implies that $f \in I(X) \subseteq C_\psi(X)$, i.e., $\mathrm{cl}_X(X \setminus Z(f))$ is pseudocompact. □

Given a topological space X, we will denote by L the set of all l-points of X and we set $N = X \setminus L$. We note that L is open and locally compact. Hence every open or closed subset of L is locally compact. Moreover every open locally compact subspace of X is contained in L.

Proposition 2.11. $C_l(X) = \bigcap_{x \in N} M_x = \{f \in C(X) : f(x) = 0, \forall x \in N\}$.

PROOF. Let $f \in C_l(X)$, $X \setminus Z(f)$ is locally compact, since it is also open, $X \setminus Z(f) \subseteq L$, so $N \subseteq Z(f)$, i.e., $f(x) = 0$, for all $x \in N$. Hence $f \in \bigcap_{x\in N} M_x$. Conversely, if $f \in \bigcap_{x\in N} M_x$, then $f(x) = 0$, for all $x \in N$, i.e., $N \subseteq Z(f)$. Hence $X \setminus Z(f) \subseteq L$, i.e., $X \setminus Z(f)$ is locally compact. □

Proposition 2.12. *If* $\mathrm{cl}_X L = X \setminus \mathrm{int}_X N$ *is locally compact (σ-compact), then* $C_{\bar{l}}(X) = C_l(X)$ $(C_\sigma(X) = C_{\bar{\sigma}}(X))$.

PROOF. If $f \in C_l(X)$ $(f \in C_\sigma(X))$, then $X \setminus Z(f) \subseteq L$ and consequently, $\mathrm{cl}_X(X \setminus Z(f)) \subseteq \mathrm{cl}_X L$. Since $\mathrm{cl}_X L$ is locally compact (σ-compact), $\mathrm{cl}_X(X \setminus Z(f))$ is so. Hence $f \in C_{\bar{l}}(X)$ $(f \in C_{\bar{\sigma}}(X))$. □

Proposition 2.13.

(a) *If* L *is σ-compact, then* $C_{l\sigma}(X) = C_l(X)$.

(b) *If* X *is second countable and* $C_{l\sigma}(X) = C_l(X)$, *then* L *is σ-compact.*

PROOF. (a) is evident. To prove (b), since L is open and X is second countable, $L = \bigcup_{n\in\mathbb{N}}(X \setminus Z(f_n))$, for $f_n \in C(X), \forall n \in \mathbb{N}$. But $X \setminus Z(f_n) \subseteq L$ implies that $f_n \in C_l(X) = C_{l\sigma}(X)$ and hence $X \setminus Z(f_n)$ is σ-compact, $\forall n \in \mathbb{N}$. This shows that L is also σ-compact.

Proposition 2.14.

1. *X is locally compact if and only if $C_{\bar{l}}(X) = C_l(X) = C(X)$, if and only if $C_{l\sigma}(X)$ is a free ideal, if and only if $C_{l\sigma}(X) = C_\sigma(X)$.*

2. *X is σ-compact if and only if $C_{\bar{\sigma}}(X) = C_\sigma(X) = C(X)$.*

3. *X is locally compact σ-compact if and only if $C_{\overline{l\sigma}}(X) = C_\infty(X)C(X) = C_{l\sigma}(X) = C(X)$.*

PROOF. The proofs of (2), the first and third parts of (1) are evident. For second part of (1), let $C_{l\sigma}(X)$ is free, then $\forall x \in X, \exists f \in C_{l\sigma}(X)$ such that $f(x) \neq 0$. Hence $x \in X \setminus Z(f) \subseteq X$. Since $X \setminus Z(f)$ is locally compact, X is a locally compact space. Conversely, let X be a locally compact space and $x \in X$. Thus there exists a compact set A in X such that $x \in \mathrm{int}_X A$. Now define $f \in C(X)$ with $f(X \setminus \mathrm{int}_X A) = \{0\}$ and $f(x) = 1$. $A_n = \{x \in X : |f(x)| \geq \frac{1}{n}\} \subseteq A$ implies that A_n is compact, for all $n \in \mathbb{N}$. Now $X \setminus Z(f) = \cup_{n=1}^{\infty} A_n$ and hence $X \setminus Z(f)$ is σ-compact. Since X is locally compact, $X \setminus Z(f)$ is also locally compact and hence $f \in C_{l\sigma}(X)$. Now $f(x) \neq 0$ shows that $C_{l\sigma}(X)$ is free.

For part (3) let X be a locally compact σ-compact space. By parts (1) and (2), $C_{\overline{l\sigma}}(X) = C_{l\sigma}(X) = C(X)$. On the other hand, Since X is locally compact σ-compact, by corollary 1.2 in [5], $C_\infty(X)$ contains a unit of $C(X)$, i.e., $C_\infty(X)C(X) = C(X)$. Conversely, if $C(X) = C_{l\sigma}(X)$, then $f = 1 \in C_{l\sigma}(X)$ implies that $X = X \setminus Z(f)$ is locally compact σ-compact. □

Proposition 2.15. *Let X be a locally compact σ-compact space. Then X is perfectly normal if and only if every open subset of X is σ-compact.*

Proof. Let A be an open subset of X. Since X is perfectly normal, there exists $f \in C(X)$ such that $X \setminus Z(f) = A$. Clearly A is locally compact σ-compact, for A is an open F_σ. Conversely, if A is an open subset of X, then A is locally compact σ-compact. By Lemma 1.1, there exists $f \in C_\infty(X)$ such that $A = X \setminus Z(f)$. Hence X is perfectly normal. □

In the following proposition, normal spaces in which the set of l-points is closed are characterized, for which the equality, $C_{\bar{l}}(X) = C_K(X)$ holds.

Proposition 2.16. *Let X be a normal space. If $C_{\bar{l}}(X) = C_K(X)$, then every closed subset of X contained in L is compact. Whenever L is closed the converse is also true, in fact if L is compact, then $C_{\bar{l}}(X) = C_K(X)$.*

Proof. First suppose that $C_{\bar{l}}(X) = C_K(X)$ and $A \subseteq L$ is closed. Since $N = X \setminus L$ is closed, $A \cap N = \emptyset$ and X is normal, There exists $f \in C(X)$ such that $f(A) = \{1\}$ and $f(N) = \{0\}$. Now $A \subseteq \{x \in X : f(x) > \frac{1}{3}\}$ and $\{x \in X : f(x) > \frac{1}{3}\}$ is a cozero-set, say $X \setminus Z(g)$. But $\mathrm{cl}_X(X \setminus Z(g)) \subseteq \{x \in X : f(x) \geq \frac{1}{3}\} \subseteq X \setminus Z(f) \subseteq X \setminus N = L$ imply that $\mathrm{cl}_X(X \setminus Z(g))$ is locally compact, i.e., $g \in C_{\bar{l}}(X)$. Since $C_{\bar{l}}(X) = C_K(X)$, $\mathrm{cl}_X(X \setminus Z(g))$ is compact. On the other hand $A \subseteq \mathrm{cl}_X(X \setminus Z(g))$ implies that A is also compact. Next suppose that every closed subset of L is compact, L is closed (compact) and $f \in C_{\bar{l}}(X)$. Then $X \setminus Z(f)$ is locally compact and so $X \setminus Z(f) \subseteq L$, hence $\mathrm{cl}_X(X \setminus Z(f)) \subseteq L$. So $\mathrm{cl}_X(X \setminus Z(f))$ is compact by our hypothesis and therefore $f \in C_K(X)$. The inclusion $C_K(X) \subseteq C_{\bar{l}}(X)$ is shown in Lemma 2.2. □

A topological space X is said to be Baire space, if the intersection of each countable family of dense open sets in X is dense. A subset A of X is called nowhere dense in X if $\mathrm{int}_X \mathrm{cl}_X A = \emptyset$. A set $A \subseteq X$ is first category in X if $A = \bigcup_{n=1}^\infty A_n$, where each A_n is nowhere dense in X. All other subsets of X are called second category in X.

It is well-known that a σ-compact space is second category (Baire) if and only if the set of l-points of X is nonempty (dense) in X. Moreover every locally compact Hausdorff space is Baire, see [12] and [4].

A nonzero ideal in a commutative ring is said to be essential if it intersects every nonzero ideal nontrivially. In [3], it is shown that a nonzero ideal E in $C(X)$ is an essential ideal if and only if $\bigcap Z[E] = \bigcap_{f\in E} Z(f)$ has an empty interior. In that article it is also shown that for a compact space X, every countable intersection of essential ideals of $C(X)$ is an essential ideal if and only if every first category subset of X is nowhere dense in X.

We conclude this section with the following propositions.

Proposition 2.17. *A σ-compact space X is a Baire space if and only if every ideal in $C(X)$ containing $C_\infty(X)$ is an essential ideal.*

PROOF. Let I be an ideal and $C_\infty(X) \subseteq I$. Then $\bigcap Z[I] \subseteq \bigcap Z[C_\infty(X)] = N$, where N is the set of all non-l-points of X. Now if X is a Baire space, the set of l-points of X is dense and hence $\operatorname{int}_X N = \emptyset$. This implies that I is essential. Conversely, let every ideal containing $C_\infty(X)$ be essential. Since $C_\infty(X) \subseteq C_l(X)$, $C_l(X)$ is also essential. Therefore $\bigcap Z[C_l(X)] = N$ has empty interior and hence the set of l-points of X is dense, i.e., X is a Baire space. □

Proposition 2.18. *A σ-compact space X is second category if and only if $C_\infty(X) \neq (0)$.*

PROOF. It is evident. □

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