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# GOLDIE DIMENSION OF RINGS OF FRACTIONS OF $C(X)$

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Abstract. It is observed that $X$ is an $F$-space if and only if $C(X)$ is locally a domain (i.e., $C(X)_P$ is a domain for each prime ideal $P$ of $C(X)$). Consequently, $X$ is an $F$-space if and only if the primary ideals of $C(X)$ in any given maximal ideal in $C(X)$ are comparable. Some of the properties of $C(X)$, where $X$ is an $F$-space, are extended to general reduced Bézout rings. It is observed that whenever $X$ is an infinite connected $F$-space, then $C(X)$ is a natural example of a non-Noetherian ring without nontrivial idempotents which is locally a domain but not a domain. We observe that the rank of a point $x \in \beta X$, in case finite, coincides with the Goldie dimension of $C(X)_{M^x}$ and give an example to show that the Goldie dimension of $C(X)_{M^x}$ is not necessarily equal to the cardinality of the set of minimal prime ideals in $M^x$. Motivated by these facts and some other appropriate ones, we define the rank of a point $x \in \beta X$ to be the Goldie dimension of $C(X)_{M^x}$. Finally, for each cardinal $\mathfrak{a}$, we show that there exists a space $X$ and a multiplicatively closed set $S$ in $C(X)$ such that the Goldie dimension of $S^{-1}C(X)$ is $\mathfrak{a}$.

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*Key words:* Goldie dimension, $F$-spaces, $P$-spaces, locally domains, reduced rings, Bézout rings, rank of a point, local cellularity, local Souslin property, flat module, SV-spaces, rings of fractions.

**1. Introduction.** One can easily see that a topological space $X$ is a $P$-space if and only if each localization of $C(X)$ at a prime ideal is a field. We observe that the counterpart of this fact holds for $F$-spaces, namely, a topological space $X$ is an $F$-space if and only if $C(X)$ is locally a domain (i.e., each $C(X)_P$ is a domain, where $P$ is any prime ideal in $C(X)$). A ring $R$ is said to have finite Goldie dimension or is finite Goldie dimensional, if there is a largest nonnegative integer $n$ with an ideal in $R$ which is a direct sum of $n$ nonzero ideals. In this case we write $\mathrm{Gdim}R = n$ (note, if there does not exist any infinite direct sum of ideals in $R$, then such an $n$ exists, see [15, p. 209]). Since we are assuming that all rings $R$ in this article are commutative with $1 \neq 0$, we infer that if $R$ is a finite Goldie dimensional ring, then $\mathrm{Gdim}R \geq 1$ (note, in case of the equality, $R$ is called a uniform ring). Clearly each domain is a uniform ring. Hence we may ask a more general question, namely, what are the topological spaces $X$ such that $C(X)$ is a locally finite Goldie dimensional ring? That is to say, each localization of $C(X)$ at prime ideals has finite Goldie dimension (resp., $C(X)$ is a locally uniform ring i.e., $\mathrm{Gdim}C(X)_P = 1$ for all prime ideals $P$ in $C(X)$). Locally domains have been extensively investigated in the literature, see for example [13, Theorem 168] and [20]. Although every Noetherian ring is clearly locally finite Goldie dimensional, it seems (at least to us) locally finite Goldie dimensional rings have not received any attention in the literature for general rings (i.e., not necessarily Noetherian rings, or reduced rings, which are the rings without nonzero nilpotent elements), to date.

It is manifest that every regular ring is locally a domain and more generally it is also easy to see that if a ring $R$ is a finite direct product of rings, each of which is a domain or a regular ring, then $R$ is locally a domain too. It is also manifest that, if $R$ has a unique minimal prime ideal (e.g., valuation rings, i.e., the rings whose ideals are totally ordered by inclusion), then it is a domain if and only it is locally a domain. In [13, Theorem 168], it is shown that a Noetherian ring is locally a domain if and only if it is a finite direct product of domains. In [14, Theorem 3.10], a similar result is proved for rings satisfying a finiteness condition weaker than being Noetherian. By what we have just noticed, if a Noetherian ring without nontrivial idempotents is locally a domain it must be a domain. In [6], it was asked whether the latter assertion remains valid if the the Noetherian assumption is dropped. Almost a decade later in [23], a rather complicated example of a non-Noetherian ring without nontrivial idempotents was constructed which is locally a domain but not a domain. As a consequence of our observation about $F$-spaces we give more examples of this phenomena. By a topological space $X$ we always mean an infinite completely regular Hausdorff space $X$ (i.e, a Tychonoff space).

Let us give a brief outline of this article which consists of three sections. Section 1, as we have already noticed, is the introduction. In Section 2, we first characterize topological spaces $X$ for which $C(X)$ is locally a domain (resp., locally uniform) and observe some useful consequences. Motivated by this characterization, we extend it to more general reduced rings. Section 3 is devoted to the Goldie dimension of the rings of fractions of $C(X)$. If $x \in \beta X$, we make a connection between $\mathrm{Gdim}C(X)_{M^x}$ and the rank of $x$. Motivated by the latter connection and some other useful facts, we naturally define the rank of a point $x \in \beta X$ to be the Goldie dimension of $C(X)_{M^x}$, when it is not necessarily finite. An example is given to show that the Goldie dimension of $C(X)_{M^x}$ and the cardinality of the set of minimal prime ideals in $M^x$ do not, in general, coincide. Given any cardinal number $\mathfrak{a}$ (infinite or finite), we show that there always exists a space $X$ and a multiplicatively closed set $S$ in $C(X)$ with $\mathrm{Gdim}S^{-1}C(X) = \mathfrak{a}$. Finally, in Sections $2, 3$, we suggest some natural questions related to the topic of this article for the sake of the interested reader. We recall that $\beta X$ is the Stone-Čech compactification of the space $X$

and for each prime ideal $P$ in $C(X)$, there exists a unique $x \in \beta X$ such that $O^x \subseteq P \subseteq M^x$, see [7, Theorem 7.15], where $M^x = \{f \in C(X) : x \in \mathrm{cl}_{\beta X} Z(f)\}$ and $O^x = \{f \in C(X) : x \in \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z(f)\}$. Finally $\{M^x : x \in \beta X\}$ is the collection of all maximal ideals of $C(X)$ and whenever $x \in X$, then $M^x$ and $O^x$ are denoted by $M_x$ and $O_x$ respectively. In fact if $M_x = \{f \in C(X) : f(x) = 0\}$ then $\{M_x : x \in X\}$ is the set of all fixed maximal ideals of $C(X)$ (note, an ideal $I$ in $C(X)$ is called fixed if $\bigcap_{f\in I} Z(f) \neq \emptyset$). The reader is referred to [13] or [25] and [7] for undefined terms and notations in algebra and topology, respectively.

## 2. Goldie dimension of localizations.

If $A$ is an ideal in a ring $R$, then $A$ is called an essential ideal in $R$ if $A$ intersects every nonzero ideal of $R$ nontrivially. A set $\{I_t\}_{t\in T}$ of nonzero ideals in a ring $R$ is said to be independent if $I_s \cap \sum_{s\neq t\in T} I_t = (0)$, i.e, $\sum_{t\in T} I_t = \oplus_{t\in T} I_t$. The Goldie dimension of a ring $R$, denoted by Gdim$R$, is the smallest cardinal number $\mathfrak{a}$ such that every independent set of nonzero ideals in $R$ has cardinality less than or equal to $\mathfrak{a}$ (note, the Goldie dimension of a ring $R$ in the literature is also called Goldie rank of $R$, the uniform dimension of $R$, rank of $R$ or simply the dimension of $R$), see [2] and [15] for various examples. The smallest cardinal number $\mathfrak{b}$ such that every family of pairwise disjoint nonempty open subsets of a space $X$ has cardinality less than or equal to $\mathfrak{b}$ is called the Souslin number or the cellularity of $X$ and is denoted by $S(X)$ or $c(X)$, see [5] and [26] for more details. It is interesting to know that Gdim$C(X) = S(X)$, see [1]. In [1] it is also observed that $|X| < \infty$ if and only if Gdim$C(X) < \infty$.

In what follows we observe some useful facts for $F$-spaces. The second part of the next result is also observed in [18, the comment preceding the proof of Theorem 1].

LEMMA 2.1. *Let $P$ be a maximal ideal in a ring $R$ and $\phi : R \to R_P$ be the natural homomorphism. If $P$ is the unique maximal ideal containing $I = Ker\phi$, then $R/I \cong R_P$. In particular, $C(X)/O^x \cong C(X)_{M^x}$.*

*Proof.* We just note that $R/I = (R/I)_{P/I} \cong R_P/IR_P = R_P$, see [25, Example 5.44] and [7, 7.12(b)]. □

Let us also cite the next result from [15, Theorems 11.43 and 11.46].

THEOREM 2.2. *The following statements are equivalent for a reduced ring $R$.*
*(a). Gdim$R = n$.*
*(b). $R$ has exactly $n$ minimal prime ideals.*
*(c). The classical ring of quotients of $R$ is the direct product of $n$ fields.*

We also recall the following result from [20, Proposition 2.1].

PROPOSITION 2.3. *The following statements are equivalent.*

*(1) Every principal ideal of $R$ is flat.*
*(2) $R_M$ is a domain for all maximal ideals $M$ of $R$.*

*(3) $R$ is reduced and every maximal ideal of $R$ contains only one minimal prime ideal of $R$.*

The statement in part (3) of the following immediate corollary first appeared in [19], see the comment following Theorem 2.1 in [8].

COROLLARY 2.4. *The following statements are equivalent for a ring $R$.*
*(1) $R_P$ is a domain for each prime ideal $P$ of $R$.*
*(2) $R_M$ is a domain for each maximal ideal $M$ of $R$.*
*(3) $R$ is reduced and every prime ideal of $R$ contains a unique minimal prime ideal.*

The next proposition is a part of our main result.

PROPOSITION 2.5. *The following statements are equivalent.*
*(a) $X$ is an $F$-space.*
*(b) $C(X)$ is locally a domain ring.*
*(c) $C(X)_M$ is a domain for every maximal ideal $M$ of $C(X)$.*
*(d) $C(X)$ is locally uniform.*

*Proof.* By the previous results (a)$\Rightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d) is trivial.

(d)$\Rightarrow$(a). Although by using Theorem 2.2 and Corollary 2.4, we can easily see that $C(X)$ is locally uniform if and only if $X$ is an $F$-space, we prefer the trivial proof which follows. We first claim that every reduced ring $R$ with $\mathrm{Gdim}R = 1$ is, in fact, a domain and this completes the proof, by Lemma 2.1. Let $ab = 0$ for $a \neq 0 \neq b$ in $R$ and get a contradiction. Take any $x \in (a)\bigcap(b)$, hence $x^2 = 0$, which implies that $x = 0$, that is to say, $(a)\bigcap(b) = 0$. Consequently $\mathrm{Gdim}R \geq 2$, which is the desired contradiction. □

We note that if $X$ is any infinite space, then there is a prime ideal $P$ in $C^*(X)$ such that $C^*(X)_P$ is never a field (note, an infinite compact space is never a $P$-space, see [7, 4K]). In contrast to the latter fact, and by using the fact that $X$ is an $F$-space if and only if $\beta X$ is so, we immediately have the following corollary.

COROLLARY 2.6. *The following statements are equivalent.*
*(a) $X$ is an $F$-space.*
*(b) $C^*(X)_P$ is a domain for each prime (resp. maximal) ideal in $C^*(X)$.*
*(c) $C^*(X)_P$ is a uniform ring for each prime (resp. maximal) ideal in $C^*(X)$.*

Considering part (d) in the preceding proposition, the characterization of spaces $X$ such that $C(X)$ is locally finite Goldie dimensional is naturally important. We will deal briefly with these spaces in the next section.

It is worthwhile to record the following fact.

EXAMPLE 2.7. In the last part of the proof of the previous proposition, we have already observed that if for each prime ideal $P$ in a reduced ring $R$, $\mathrm{Gdim}R_P = 1$, then $R$ is locally a domain. We should emphasize here that there are commutative rings $R$ such that Gdim$R_P = 1$ for all prime ideals $P$ of $R$ (i.e., are locally uniform), but no $R_P$ is a domain. To see this, let $p$ be any prime number, then consider the ring $R = \frac{\mathbb{Z}}{(p^r)}$, where $r \geq 2$ is an integer (note, $R$ has only $r-1$ nonzero ideals and they form a chain and it has only one prime ideal which is $P = \frac{(p)}{(p^r)}$). Clearly $R = R_P$ and Gdim$R = 1$, but $R$ is not a domain. It is also noteworthy to mention that if $T$ is the ring which is the product of $n$ copies of $R$, where $n$ is any positive integer, then Gdim$T = n$ and Gdim$T_P = 1$ for all prime ideals $P$ of $T$.

Using Proposition 2.5, along with the fact that the localization of a Bézout ring is Bézout and applying [13, Theorem 63], we immediately have the interesting result which follows, see also the comment following Corollary 2.5 in [11].

COROLLARY 2.8. *A topological space $X$ is an $F$-space if and only if the localization of $C(X)$ (resp. $C^*(X)$) at any prime ideal of $C(X)$ (resp. $C^*(X)$) is a valuation domain.*

Using [7, Theorem 7.13] one can slightly strengthen [7, Theorem 7.15] as follows.

REMARK 2.9. Every primary ideal $Q$ in $C(X)$ contains $O^x$ for a unique $x \in \beta X$, and $M^x$ is the unique maximal ideal containing $Q$.

The above two results together with [25, Theorem 5.37], immediately yield the following stronger property for $F$-spaces with contrast to the property that the prime ideals in a given maximal ideal in $C(X)$, where $X$ is an $F$-space, form a chain, see [7, Theorem 14.25(2)].

COROLLARY 2.10. *A topological space $X$ is an $F$-space if and only if the primary ideals in any given maximal ideal in $C(X)$ (resp. $C^*(X)$ ) form a chain.*

Using Propositions 2.3, 2.5, Corollary 2.4 and [24, Corollary 3.49], the next corollary is now immediate.

COROLLARY 2.11. *A topological space $X$ is an $F$-space if and only if every ideal in $C(X)$ is flat.*

With regard to the existence of non-Noetherian rings without nontrivial idempotents which are locally domains, but not domains, see [23] for such a phenomena. To give more examples, let us recall the simple and well-known fact that $X$ is a connected space if and only if $C(X)$ has no nontrivial idempotents, see [7, 1B]. Let us also emphasize that if $X$ is any space, then $C(X)$ is never Noetherian or a domain, see [2, Remark 2.12]. Consequently, whenever $X$ is a connected $F$-space (e.g., $X = \beta\mathbb{R}^+ \setminus \mathbb{R}^+$), see [7, p. 211], then by Proposition 2.5, $C(X)$ provides us with the natural examples of these phenomena.

So far we have noticed that locally domains and $C(X)$, where $X$ is an $F$-space, enjoy some common properties. Therefore in order to present our last result in this section, which is in fact, an algebraic characterization of $F$-spaces, let us first introduce some notations in general rings, similar to their counterparts in $C(X)$.

Let $S$ be a multiplicatively closed set in $R$ and put $O_S = \{r \in R : rs = 0 \text{ for some } s \in S\}$, hence $O_S = ker\varphi$, where $\varphi : R \to S^{-1}R$ is the natural ring homomorphism (i.e., $\varphi(r) = \frac{r}{1}$). Clearly $O_S$ is an ideal of $R$ and it is prime (resp. semiprime) in $R$ if and only if $S^{-1}R$ is a domain (resp. a reduced ring). If $S = R \setminus P$, where $P$ is a prime ideal contained in $R$, then similar to the convention $S^{-1}R = R_P$, we may also put $O_S = O_P$ and it is contained in every prime ideal which is in $P$. Clearly, in the latter case if $R$ is reduced, then $P$ is a minimal prime ideal if and only if $O_S = P$, see [12, Corollary 2.2]. As we have already noticed for an element $x \in \beta X$, we have $f \in O^x$ if and only if $fg = 0$ for some $g \notin M^x$, that is to say, $O^x = O_{M^x}$, by our notation. Clearly, by Proposition 2.5 and [7, Theorem 14.25], $X$ is an $F$-space if and only if $O_P$ is prime for each prime ideal $P$ of $C(X)$.

We are now ready to present our main result in this section (which is in fact the algebraic characterization of $F$-spaces) and although some of its statements are already mentioned above for reduced rings, we repeat them for the record.

Theorem 2.12. *Let $R$ be a reduced Bézout ring. The following statements are equivalent.*

*(1) $O_P$ is a prime ideal for each prime ideal $P$ of $R$ (e.g., $R = C(X)$, where $X$ is an $F$-space).*
*(2) $R$ is locally a domain.*
*(3) Every prime ideal $P$ of $R$ contains a unique minimal prime ideal $Q$ such that $O_P = O_Q = Q$.*
*(4) $R_M$ is a domain for each maximal ideal $M$ of $R$.*
*(5) Every maximal ideal $M$ of $R$ contains a unique minimal prime ideal $P$ with $O_M = O_P = P$.*
*(6) $R_P$ is a valuation domain for each prime ideal $P$ of $R$.*
*(7) Primary ideals contained in each prime ideal of $R$ form a chain.*
*(8) $R$ is a locally uniform ring.*
*(9) $R$ is a locally finite Goldie dimensional ring and the set of zero divisors of $\frac{R}{O_P}$ is an ideal.*
*(10) Every ideal of $R$ is flat.*

*Proof.* (1)⇒(2) It is evident by the precedent comment.
(2)⇒(3). It is manifest that every prime ideal of $R$ contains a unique minimal prime ideal. Hence by the preceding comment, it suffices to show that $O_P$ is prime (note, $O_P$ is in every prime ideal which is in $P$). Clearly $\frac{R}{O_P}$ is embeddable in $R_P$ and we are done.
(3)⇒(4)⇒(5). It is evident by Proposition 2.3, Corollary 2.4 and the comment preceding the theorem.
(5)⇒(6). It is clear by Corollary 2.4 and [13, Theorem 63].
(6)⇒(7). It is evident by [25, Theorem 5.37(iv)].
(7)⇒(8). We first note that each prime ideal contains a unique minimal prime ideal. Thus for each prime ideal $P$ of $R$, $R_P$ contains a unique minimal prime ideal which must be zero, for $R_P$ is reduced, hence $R_P$ is a domain and we are done (This is also a consequence of Proposition 2.3 and Corollary 2.4).

(8)⇒(9). As we observed in the proof of (d)⇒(a) in Proposition 2.5, every reduced ring whose Goldie dimension is 1 is a domain. Thus $R$ is locally a domain and it remains to be shown that the set of zero divisors of $\frac{R}{O_P}$ is an ideal for each prime ideal $P$ of $R$. But this set actually consists of the zero element, for in fact $O_P$ is a prime ideal (note, the latter quotient ring is embeddable in $R_P$) and we are through.

(9)⇒(10). We first show that $R$ is locally domain. To see this, for each prime ideal $P$ of $R$ we observe that $R_P$ has only finitely many minimal prime ideals. Hence we may assume $P$ has only $n$ minimal prime ideals, say $\{P_1, P_2, ..., P_n\}$. Clearly $n = \text{Gdim}R_P$, by Theorem 2.2. Since $\frac{R}{O_P}$ is reduced, we infer that its set of zero divisors is the union of its minimal prime ideals, by [13, P 63, Example 13]. Since $\frac{R}{O_P}$ is a subring of $R_P$, we infer that each minimal prime ideal of $\frac{R}{O_P}$ is a contraction of a minimal prime ideal of $R_P$, by [13, P 41, Example 1] (note, in fact in [13, P 41, Example 1] it is only claimed that if $R \subseteq T$ are rings and $Q$ is a minimal prime ideal of $R$, then $Q = R \cap Q_1$ for some prime ideal $Q_1$ in $T$). But one can see easily that $Q_1$ can be assumed to be a minimal prime ideal in $T$. To see this, let $Q_2 \subseteq Q_1$ be a minimal prime ideal in $T$, hence $Q_2 \cap R$ is a prime ideal in $R$ and since $Q_2 \cap R \subseteq Q$, we infer that $Q_2 \cap R = Q$, and we are done. Consequently, $\frac{R}{O_P}$ has exactly $n$ minimal prime ideals, which are $\frac{P_1}{O_P}, \frac{P_2}{O_P}, \dots, \frac{P_n}{O_P}$. Since $\frac{R}{O_P}$ is a reduced ring, we infer that its set of zero divisors, say $\frac{Z}{O_P}$, is the union of the above minimal prime ideals. But by our assumption, $\frac{Z}{O_P}$ is an ideal, hence by the prime avoidance lemma, see [25, Theorem 3.61] or [13, Theorem 81], $\frac{Z}{O_P}$ is contained in one of the minimal prime ideals of $\frac{R}{O_P}$. Thus the number of the minimal prime ideals of $\frac{R}{O_P}$ reduces to one, i.e., $R_P$ has only one minimal prime ideal which causes $R_P$ to become a domain (note, $R_P$ is a reduced ring) and we are done. Now by invoking Theorem 2.3, we infer that each principal ideal and therefore each finitely generated ideal of $R$ is flat. Finally, by [24, Corollary 3.49], each ideal of $R$ is flat and we are through.

(10)⇒(1). By Proposition 2.3 and Corollary 2.4, $R$ is locally a domain, hence $O_P$ is a prime ideal for each prime ideal $P$ of $R$ (note, $\frac{R}{O_P}$ is embeddable in $R_P$) and this completes the proof. □

In the proof of (9)⇒(10), we have actually shown that when $R$ is reduced and $\text{Gdim}R_P$ is finite, then $\text{Gdim}\frac{R}{O_P} = GdimR_P$. Motivated by this and the fact that the operation of extension of ideals from $R$ to $S^{-1}R$ acts naturally on the finite sum of ideals and also on the finite intersection of ideals in $R$, see [25, Lemma 5.31], we record the following general result (which seems to have been overlooked in the literature), even when the ring is not necessarily reduced and the dimensions are also not necessarily finite.

PROPOSITION 2.13. *Let $S$ be a multiplicatively closed set in a ring $R$. Then $Gdim\frac{R}{O_S} = GdimS^{-1}R$.*

Finally, our investigation in this section leads us to the following four questions.

Questions.

1. What are the Noetherian rings $R$ such that there exists an integer $n$ with $\mathrm{Gdim} R_P \le n$ for each prime ideal $P$ of $R$ (or equivalently, $\mathrm{Gdim}\frac{R}{O_P} \le n$ for all prime ideals $P$ of $R$)? Noetherian rings which are direct product of domains have this property. More generally, every Noetherian reduced ring has this property.
2. Characterize general rings which are locally finite Goldie dimensional (equivalently, for each prime ideal $P$ of $R$, $\frac{R}{O_P}$ has finite Goldie dimension).
3. Characterize rings $R$ in the second question with a finite upper bound on $\mathrm{Gdim} R_P$ for all prime ideals $P$ of $R$ (note, any finite direct product of $C(X)$, where $X$ is an infinite $F$-space, has this property. Also any reduced finite Goldie dimensional ring has this property too and we should emphasize that the former ring has infinite Goldie dimension). This question can also be generalized by replacing finite upper bound with a fixed infinite upper bound, for example $\aleph_0$ or $\aleph_1$ (equivalently, characterize rings $R$ such that $\mathrm{Gdim}\frac{R}{O_P} \le \lambda$ for all prime ideals $P$ of $R$, where $\lambda$ is a fixed cardinal number). In particular, we are interested in those rings for which this infinite upper bound is attained (i.e., $\mathrm{Gdim} R_P$ is equal to this upper bound for some prime ideal $P$ of $R$).
4. Motivated by Proposition 2.13, it is natural to ask for the characterization of all the ring extensions $R \subseteq T$ such that $\mathrm{Gdim} R = \mathrm{Gdim} T$. By Proposition 2.13, it is clear that whenever $T$ is an overring of $R$ (i.e., $R \subseteq T \subseteq Q(R)$, where $Q(R)$ is the total (classical) quotient ring of $R$, i.e., $Q(R) = S^{-1}R$, where $S$ is the set of all non-zero divisors of $R$), then $\mathrm{Gdim} R = \mathrm{Gdim} T = \mathrm{Gdim} Q(R)$.

**3. The rank of a point vs. Goldie dimension.** For each $x \in \beta X$, $rk(x)$ denotes the number of minimal prime ideals contained in $M^x$, if the set of all such minimal prime ideals is finite, and $rk(x) = \infty$ otherwise. The number $rk(x)$ is called the rank of $x$. The notion of "rank of a point" is first introduced and studied in [9]. The following result first appears in [9] for compact Hausdorff spaces and later in [16] for completely regular Hausdorff spaces.

Theorem 3.1. *Let $X$ be a topological space. A point $x \in X$ has finite rank $n \ge 2$ if and only if there is a collection of $n$ pairwise disjoint cozerosets such that $x$ is in the closure of each of these cozerosets and there is no larger such collection.*

Using Theorem 2.2, we observe trivially that whenever $rk(x)$ is finite, it is in fact the Goldie dimension of the reduced ring $C(X)_{M^x}$. Although, by what we have already observed, the first part of the next theorem follows accordingly, since it immediately yields Theorem 3.1, we present a direct proof in the context of $C(X)$ for the sake of completeness. Before doing this, the following definition and the next lemma are needed.

Definition. A collection $F = \{C_i : i \in I\}$ of nonempty cozerosets in a space $X$ is said to have a disjoint refinement if there exists a collection of disjoint nonempty cozerosets, say $E = \{B_i : i \in I\}$, with $B_i \subseteq C_i$ for each $i \in I$ and $E$ is called a disjoint refinement of $F$.

LEMMA 3.2. *Let $x \in X$ and $F = \{X \setminus Z(f_i) : i = 1, 2, ..., n\}$ be a collection of nonempty cozerosets whose closures contain $x$ and $f_i f_j \in O_{M_x}$ for all $i \neq j$. Then $F$ has a disjoint refinement with the latter properties.*

*Proof.* By our assumption $f_i \notin O_x$ for all $i$ and for all $j \neq i$, there must exists $g_{ij} \notin M_x$ with $f_i f_j g_{ij=0}$. Now by putting $g = \prod_{i,j=1}^{n} g_{ij}$, we infer that $g \notin M_x$ and hence $gf_i \notin O_x, \forall i = 1, 2, \ldots, n$ (note, otherwise $f_i \in O_x$ which is absurd, for by [7, 7.15], no prime ideal containing $O_x$ can contain $g$, for $g \notin M_x$, hence $f_i$ is in every prime ideal containing $O_x$ and we are done). This implies that $x \in \mathrm{cl}_X(X \setminus Z(gf_i))$ and $(X \setminus Z(gf_i)) \cap (X \setminus Z(gf_j)) = \emptyset$. Consequently, $\{X \setminus Z(gf_i) : i = 1, 2, ..., n\}$ is a collection of $n$ pairwise disjoint cozerosets with $x$ in their closures (note, it is trivial to see that $gf_i gf_j = 0 \in O_{M_x}$ and $X \setminus Z(gf_i) \subseteq X \setminus Z(f_i)$). This completes the proof. □

THEOREM 3.3. *If $x \in \beta X$ has a finite rank, then Gdim$C(X)_{M^x} = rk(x)$. Moreover, the latter equality yields Theorem 3.1.*

*Proof.* Let $rk(x) = n$ and $\{P_1, P_2, \ldots, P_n\}$ be the set of all minimal prime ideals contained in $M^x$. Since $\bigcap_{j \neq i} P_j \nsubseteq P_i$, there exists $f_i \in \bigcap_{j \neq i} P_j \setminus P_i$. Since $O^x = \bigcap_{i=1}^{n} P_i$, $f_i \notin O^x$, $\forall i = 1, 2, \ldots, n$. This means that $\frac{f_i}{1} \neq 0$, $\forall i = 1, 2, \ldots, n$, by Lemma 2.1. But $f_i f_j \in O^x = \bigcap_{i=1}^{n} P_i$, which means that $\{(\frac{f_i}{1}) : 1 \leq i \leq n\}$ is an independent set of nonzero ideals of $C(X)_{M^x}$, by Lemma 2.1, hence Gdim$C(X)_{M^x} \geq n$. Now suppose that $\mathfrak{B} = \{B_1, B_2, \ldots, B_{n+1}\}$ is an independent set of nonzero ideals of $C(X)_{M^x}$ and take $0 \neq \frac{f_i}{1} \in B_i$, $i = 1, 2, \ldots, n+1$. By Lemma 2.1, $f_i \notin O^x$, $i = 1, 2, \ldots, n+1$ and hence $f_i$ is not contained in some minimal prime ideal in $M^x$. Without loss of generality, we may assume that $f_i \notin P_i$. One of the $P_i$'s does not contain at least two of the $f_j$'s, let $f_i, f_j \notin P_i$. But by the independence of the set $\mathfrak{B}$, we have $\frac{f_i}{1}\frac{f_j}{1} = 0$ which means that $f_i f_j \in O^x \subseteq P_i$, a contradiction. Therefore Gdim$C(X)_{M^x} = n$.

Now to prove Theorem 3.1, let us assume that $x \in X$ and $rk(x) = n$. Hence there is an independent set $\{B_1, \ldots, B_n\}$ of nonzero ideals in $C(X)_{M_x}$ and $n$ is the largest integer with this property. Without loss of generality, we may take each $B_i = (\frac{f_i}{1})$ to be principal, then we have $f_i \notin O_x$, $1 \leq i \leq n$ and $f_i f_j \in O_x = O_{M_x}$, $\forall\ i \neq j$, by Lemma 2.1. Now $f_i \notin O_x$ implies that $x \in \mathrm{cl}_X(X \setminus Z(f_i))$. Consequently, by the previous lemma, the collection $\{X \setminus Z(f_i) : i = 1, 2, ...., n\}$ has a disjoint refinement with $x$ in the closure of each member of this disjoint refinement . Therefore it remains to be shown that there is no larger collection of disjoint cozerosets whose closures contain $x$. This is also evident, for if there is such a collection, then there is an independent collection of more than $n$ principal ideals, say $\{(f_1), (f_2), \ldots, (f_{n+1})\}$ in $M_x$ such that no $f_i$ is in $O_x$. Hence we have a collection $\{(\frac{f_1}{1}), (\frac{f_2}{1}), \ldots, (\frac{f_{n+1}}{1})\}$ of $n+1$ independent nonzero ideals in $C(X)_{M_x}$, that is to say that Gdim$C(X)_{M_x}$ is greater than $n$, which is absurd by the first part, and we are done. Conversely, similarly to what we have just shown, the existence of a largest integer $n$ with a family consisting of $n$ pairwise disjoint cozerosets such that $x$ is in their closures, implies that the Goldie dimension of the ring $C(X)_{M_x}$ is $n$, and we are through by the first part of the theorem. □

Let $x \in X$, then $X$ is said to have local Souslin property at $x$ if whenever $F = \{X \setminus Z(f_i) : i \in I\}$ is a collection of cozerosets with $x$ in their closures and $f_i f_j \in O_{M_x}$, it has a disjoint refinement with the former property. We have already noticed, by Lemma 3.2 and Theorem 3.3, that if $x \in X$ has finite rank, then $X$ has local Souslin property at $x$. Motivated by the latter results, for any $x \in X$, the smallest cardinal number $\mathfrak{a}$ such that whenever $F = \{X \setminus Z(f_i) : i \in I\}$ is a collection of cozerosets in $X$, with $x$ in their closures and $f_i f_j \in O_{M_x}$ for all $i \neq j$, then $|I| \leq \mathfrak{a}$, is called the local cellularity at $x$ and is denoted by $c_l(x)$. If $rk(x)$ is finite, we have already seen that $rk(x) = c_l(x) = \mathrm{Gdim}C(X)_{M^x}$. In what follows we prove a more general result, which shows that the Goldie dimension of a localization of $C(X)$ may be an infinite cardinal. Although, its proof is more or less the same as the above proof, we present it for the sake of the reader. Before giving the result, let us inform the reader that whenever $x \in \beta X$, then the previous result and the one which follows, pave the way for us, to define the rank of the point $x$ (whether finite or infinite) to be the Goldie dimension of $C(X)_{M^x}$, that is to say, we define $rk(x) = \mathrm{Gdim}C(X)_{M^x}$.

Proposition 3.4. *Let $X$ be a topological space. Then for any $x \in X$ we have $GdimC(X)_{M_x} = c_l(x)$.*

*Proof.* Let $\mathcal{C} = \{X \setminus Z(f_i) : i \in I\}$ be a collection of cozerosets with the property that $x \in \mathrm{cl}_X(X \setminus Z(f_i))$ for all $i \in I$ and $f_i f_j \in O_{M_x}$ $\forall i \neq j$ in $I$. Hence $f_i \notin O_x$, $\forall i \in I$. Consequently $\{(\frac{f_i}{1}) : i \in I\}$ is an independent set of nonzero ideals in $C(X)_{M_x}$, see Lemma 2.1, and the comment preceding Theorem 2.12. Hence $\mathrm{Gdim}C(X)_{M_x} \geq |I|$, which means that $\mathrm{Gdim}C(x)_{M^x} \geq c_l(x)$. Now let $\{(\frac{f_i}{1}) : i \in I\}$ be an independent collection of nonzero principal ideals in $C(X)_{M_x}$. To prove that $c_l(x) \geq \mathrm{Gdim}C(X)_{M_x}$, it suffices to show that $c_l(x) \geq |I|$. Since the previous collection is independent, we immediately infer that for all $i, j \in I$ we have $f_i \notin O_x$ and $f_i f_j \in O_x = O_{M_x}$ (note, again we are using Lemma 2.1, and the comment preceding Theorem 2.12). Hence $\{X \setminus Z(f_i) : i \in I\}$ is a collection of cozerosets with the property that $x \in \mathrm{cl}_X(X \setminus Z(f_i))$ for each $i$ and $f_i f_j \in O_x = O_{M_x}$. Now by the definition of $c_l(x)$ in the precedent comment, we infer that $|I| \leq c_l(x)$ and this completes the proof. □

The following result which shows that the local cellularity at non-isolated points in a metric space is at least uncountable, is needed. We first recall the well-known fact that for each infinite countable set $E$, there exists $\mathcal{F} \subseteq \mathcal{P}(E)$ consisting of infinite subsets of $E$ such that $|\mathcal{F}| = 2^{\aleph_0}$ and $A \cap B$ is a finite set for all $A, B \in \mathcal{F}$, see [7, 6Q, 6S], see also [2, Remark 2.7].

PROPOSITION 3.5. *Suppose that $X$ is a first countable space whose closed sets are zerosets (e.g., metric spaces) and let $x \in X$ be a non-isolated point. Then $GdimC(X)_{M_x} \geq 2^{\aleph_0}$. In particular, if $X$ is a connected metric space, then $GdimC(X)_M \geq 2^{\aleph_0}$ for every fixed maximal ideal $M$ of $C(X)$.*

*Proof.* Since $x$ has a countable base, we may consider $\mathfrak{B}_x = \{G_n : n \in \mathbb{N}\}$ as a base at $x$ with $G_1 \supsetneqq \overline{G_2} \supseteq G_2 \supsetneqq \cdots \supsetneqq \overline{G_n} \supseteq G_n \cdots$. Now Let $H_n = G_n \setminus \overline{G_{n+1}}$ for each $n \in \mathbb{N}$ and put $E = \{H_n : n \in \mathbb{N}\}$. Consequently, by the preceding comment, there exists $\mathcal{F} \subseteq \mathcal{P}(E)$ consisting of infinite subsets of $E$ such that $|\mathcal{F}| = 2^{\aleph_0}$ and $\mathcal{A} \cap \mathcal{B}$ is a finite set for all $\mathcal{A}, \mathcal{B} \in \mathcal{F}$. Now for each $\mathcal{A} \in \mathcal{F}$, we consider the set $\mathcal{G}_{\mathcal{A}} = \bigcup \mathcal{A}$ and put $\mathcal{C} = \{\mathcal{G}_{\mathcal{A}} : \mathcal{A} \in \mathcal{F}\}$. It is manifest that $|\mathcal{C}| = 2^{\aleph_0}$. By our hypothesis, we infer that each $\mathcal{G}_{\mathcal{A}}$ is a cozeroset, say $X \setminus \mathcal{G}_{\mathcal{A}} = Z(f_{\mathcal{A}})$. By our definition of $H_n$'s, it is easy to see that $x \in \mathrm{cl}_X \mathcal{G}_{\mathcal{A}}$. We claim that $Z(f_{\mathcal{A}}) \cup Z(f_{\mathcal{B}})$ contains a neighborhood of $x$, for all $\mathcal{A}, \mathcal{B} \in \mathcal{F}$, hence $f_{\mathcal{A}} f_{\mathcal{B}} \in O_x$. To see this, we just observe that $\mathcal{G}_{\mathcal{A}} \cap \mathcal{G}_{\mathcal{B}}$ is in fact either empty or a finite union of $H_n$'s. Hence by our definition of $H_n$'s, there exists a neighborhood, say $G_x$ of $x$ such that $G_x$ does not intersect $\mathcal{G}_{\mathcal{A}} \cap \mathcal{G}_{\mathcal{B}}$. Accordingly, $G_x \subseteq Z(f_{\mathcal{A}}) \cup Z(f_{\mathcal{B}})$, i.e., $f_{\mathcal{A}} f_{\mathcal{B}} \in O_x$. Finally by invoking Proposition 3.4, we are done. □

The following remark shows that the rank of a point in a topological space is not necessarily equal to the cardinality of the set of minimal prime ideals in the corresponding fixed maximal ideal.

REMARK 3.6. Let $X$ be an infinite countable discrete space and $X^* = X \cup \{y\}$ be its one-point compactification. Then $\mathrm{Gdim}C(X^*)_{M_x} = 1$ for all $x \in X$ and $\mathrm{Gdim}C(X^*)_{M_y} = 2^{\aleph_0}$ (note, $\mathrm{Gdim}C(X^*)_{M_y} \geq 2^{\aleph_0}$, by the above proposition). By [7, 14G(5)], we observe that the cardinality of the set of minimal prime ideals in $M_y$ is $2^{\mathfrak{c}}$ (note, for each $x \in X$, $M_x$ is a minimal prime ideal). Moreover if $X$ is any infinite discrete space (not necessarily countable) and $X^* = X \cup \{y\}$ is its one-point compactification, then $\mathrm{Gdim}C(X^*)_{M_x} = \mathrm{Gdim}\frac{C(X^*)}{O_x} = 1$ for all $x \in X$ and $\mathrm{Gdim}C(X^*)_{M_y} = \mathrm{Gdim}\frac{C(X^*)}{O_y} \geq |X|$. To see this, we should observe that $O_y$ is the socle of $C(X^*)$ (i.e., $O_y = C_F(X^*)$) and $|X| = \mathrm{Gdim}C(X) = \mathrm{Gdim}C(X^*)$, see [2, Theorem 2.5(3)], [1, Theorem 2.2] and Lemma 2.1.

Concerning the comment which follows Corollary 2.6, we must admit that part (d) and part (9) in Proposition 2.5 and Theorem 2.12, respectively, might lead us on, to guess that "$X$ is an $F$-space if and only if $C(X)$ is locally finite Goldie dimensional". But this is not always true and in fact there are many examples of spaces $X$ such that $C(X)$ is locally finite Goldie dimensional, but not necessarily locally a domain. First we recall that a space $X$ is an $FMP$-space at $p \in \beta X$ if $rk(p)$ is finite and $X$ is called an $FMP$-space if it is an $FMP$-space at every point of $\beta X$. Hence by Theorem 2.2, $X$ is an $FMP$-space if and only if every localization of $C(X)$ at prime ideals has finite Goldie dimension. Clearly every $F$-space is an $FMP$-space and for examples of $FMP$-spaces which are not necessarily $F$-spaces, see [11, Example 2.9] and [22, Proposition 4.2]. Note that [11, Example 2.9] gives a compact $SV$-space (a space $X$ such that $\frac{C(X)}{P}$ is a valuation domain for each prime ideal of $C(X)$) which is not an $F$-space. We also remind the reader that every $SV$-space is an $FMP$-space, see [10, Note added in proof], and [17, Theorem 1.1(2)]. We should also bring to the attention of the reader that [9, Corollary 4.2.1] and [17, Theorem 1.1(2)] in fact show that whenever $X$ is an $SV$-space, then there is an integer $n$ such that $\text{Gdim}C(X)_P \leq n$ for all prime ideals in $C(X)$. Consequently, $C(X)$ in this case is a partial answer to the third question in the preceding section. Motivated by this and by what we have already observed earlier, we present the next two questions.

Questions.

1. What are the topological spaces $X$ such that $\text{Gdim}C(X)_P \leq \lambda$ for all prime ideals $P$ in $C(X)$, where $\lambda$ is a given cardinal number (equivalently, for each prime ideal $P$ in $C(X)$, $\text{Gdim}\frac{C(X)}{O_P} \leq \lambda$)? For example, let $\lambda \in \mathbb{N}$ or $\lambda = \aleph_0$ or $= \aleph_1$. In particular, when $\lambda$ is infinite we are interested in topological spaces $X$ for which this $\lambda$ is attained (i.e., there exists a prime ideal $P$ in $C(X)$ with $\text{Gdim}C(X)_P = \lambda = \text{Gdim}\frac{C(X)}{O_P}$).
2. What are the topological spaces $X$ such that for each $x \in X$ we have $rk(x) = \text{Gdim}C(X)_{M^x} = |\text{Min}(C(X)_{M^x})|$, where for any ring $R$, $\text{Min}(R)$ is the set of all minimal prime ideals of $R$?

For each subset $A$ of $\beta X$ put $S_A = C(X) \setminus \bigcup_{x\in A} M^x$. Clearly $S_A$ is a multiplicatively closed set which is saturated (i.e., $fg \in S_A$ if and only if $f \in S_A$ and $g \in S_A$). In light of the Gelfand-Kolmogoroff theorem, see [7, p.102], one can easily see that $S_A = \{g \in C(X) : A \cap \text{cl}_{\beta X} Z(g) = \emptyset\}$. Let us also recall that for each subset $A$ of $\beta X$, $M^A = \{f \in C(X) : A \subseteq \text{cl}_{\beta X} Z(f)\} = \bigcap_{x\in A} M^x$ and $O^A = \{f \in C(X) : A \subseteq \text{int}_{\beta X}\text{cl}_{\beta X} Z(f)\} = \bigcap_{x\in A} O^x$. Finally, let $\varphi_A : C(X) \to S^{-1}C(X)$ be the natural ring homomorphism (i.e., $\varphi(f) = \frac{f}{1}$). The next lemma, which is now in order, is also needed.

Lemma 3.7. *Let $A$ be a closed subset of $\beta X$. Then $O^A = O_{S_A} = ker\varphi_A$.*

*Proof.* Let $f \in ker\varphi_A$, i.e., $\frac{f}{1} = 0$, hence there exists $g \in S_A$ such that $fg = 0$. By what we have already observed above, $A \cap \text{cl}_{\beta X} Z(g) = \emptyset$, hence $A \subseteq \beta X \setminus \text{cl}_{\beta X} Z(g)$. But $Z(f) \cup Z(g) = X$ implies that $\text{cl}_{\beta X} Z(f) \cup \text{cl}_{\beta X} Z(g) = \beta X$, see [7, Theorem 6.5(IV)] which, in turn, implies that $\beta X \setminus \text{cl}_{\beta X} Z(g) \subseteq \text{int}_{\beta X}\text{cl}_{\beta X} Z(f)$. Therefore $A \subseteq \text{int}_{\beta X}\text{cl}_{\beta X} Z(f)$, i.e., $f \in O^A$. Conversely, let $f \in O^A$. We first recall that, in [3] and [4], it is shown that for all closed subsets $A$ of $\beta X$, $O^A$ are exactly the pure ideals of $C(X)$ (note that an ideal $I$ in a ring $R$ is said to be pure if for every $x \in I$, there is $y \in I$ such that $x = xy$, see also [24]). Since $O^A$ is a pure ideal, there exists $g \in O^A$ such that $f = fg$ or $f(1-g) = 0$. But $\text{cl}_{\beta X} Z(g) \cap \text{cl}_{\beta X} Z(1-g) = \emptyset$, see [7, Theorem 6.5(III)], implies that $A \subseteq \text{cl}_{\beta X} Z(g) \subseteq \beta X \setminus \text{cl}_{\beta X} Z(1-g)$ and therefore $A \cap \text{cl}_{\beta X} Z(1-g) = \emptyset$, i.e., $1-g \in S_A$, so $\frac{f}{1} = 0$. □

The next example shows that for each $n \in \mathbb{N}$, there exist a topological space $X$ and a multiplicatively closed set $S$ such that $\text{Gdim}S^{-1}C(X) = n$.

EXAMPLE 3.8. Let $A = \{x_1, x_2, \ldots, x_n\}$ be a set of $F$-points in the space $X$ and $S = C(X)\setminus\bigcup_{x\in A} O^x$. Since $O^x$, for each $x \in A$, is a minimal prime ideal, $S^{-1}C(X)$ has exactly $n$ minimal prime ideals and hence by Theorem 2.2 and Proposition 2.13, $\mathrm{Gdim} S^{-1}C(X) = \mathrm{Gdim}\frac{C(X)}{O_S} = n$. We may also select $n$ minimal prime ideals $P_1, \ldots, P_n$ in any reduced ring $R$ and consider $S = R\setminus\bigcup_{i=1}^{n} P_i$ and again by Theorem 2.2 and Proposition 2.13, we have $\mathrm{Gdim} S^{-1}R = \mathrm{Gdim}\frac{R}{O_S} = n$.

Let us recall that if $A \subseteq X$, then $O^A = O_A = \{f \in C(X) : A \subseteq \mathrm{int}_X Z(f)\}$ and $S_A = C(X) \setminus \bigcup_{x\in A} M_x$. Let us also recall that if $\mathrm{Gdim}R = \lambda$ and $\lambda$ is not an inaccessible cardinal, then there is an independent collection of nonzero ideals in $R$ whose cardinal is $\lambda$ (i.e., $\mathrm{Gdim}R = \lambda$ is attained), see [2, Remark 2.10]. We are now ready for the next result in this section.

THEOREM 3.9. *Let $\{G_i : i \in I\}$ be a collection of disjoint open sets in $X$ and for each $i \in I$, let $x_i \in G_i$ be such that $X$ has local Souslin property at $x_i$. Suppose that $A = \{x_i : i \in I\}$ is a closed set in $\beta X$ and for each $i \in I$, $\mathrm{Gdim}C(X)_{M_{x_i}}$ is attained (e.g., if $A \subseteq X$ is just a finite set of points whose ranks are finite), then $\mathrm{Gdim}S_A^{-1}C(X) = \sum_{i\in I} \mathrm{Gdim}C(X)_{M_{x_i}} = \sum_{i\in I} c_l(x_i)$.*

*Proof.* Let $\mathrm{Gdim}C(X)_{M_{x_i}} = \lambda_i$ for all $i \in I$ and $\{(\frac{f_{ij}}{1}) : j \in J_i\}$, where $|J_i| \leq \lambda_i$, be an independent collection of nonzero ideals in $C(X)_{M_{x_i}}$ (note, with no loss of generality, we may suppose that $J_i \cap J_k = \emptyset$, for all $i \neq k$). For each $i \in I$ we may also consider a cozeroset $X \setminus Z(g_i)$ with $x_i \in X \setminus Z(g_i) \subseteq G_i$ and by our assumption, these cozerosets are mutually disjoint. Clearly $\frac{h}{1} \neq 0$ and $\frac{h}{1}\frac{k}{1} = 0$ in $C(X)_{M_{x_i}}$ imply that $h \notin O_{x_i}$ and $hk \in O_{x_i} = O_{M_{x_i}}$, by Lemma 2.1 and the comment preceding Theorem 2.12. Hence the collection $\{X \setminus Z(f_{ij}) : j \in J_i\}$ has the property that $x_i \in \mathrm{cl}_X(X \setminus Z(f_{ij}))$ and $f_{ij}f_{ik} \in O_{M_{x_i}}$, for $j \neq k$. Since $X$ has local Souslin property at each $x_i$, we may, without loss of generality, assume that the latter collection is, in fact, a collection of disjoint cozerosets for each $i \in I$. Now consider the collection $\{(\frac{g_i f_{ij}}{1}) : i \in I, j \in \bigcup_{i\in I} J_i = J\}$ of ideals in $S_A^{-1}C(X)$. We claim that this collection is independent. To this end, we first note that $\frac{g_i f_{ij}}{1} \neq 0$ for all $i \in I$, $j \in J$, for otherwise by Lemma 3.7, $g_i f_{ij} \in O^A = O_A$, i.e., $A \subseteq \mathrm{int}_X Z(g_i f_{ij})$, but $x_i \in A$, $x_i \in \mathrm{cl}_X(X \setminus Z(g_i f_{ij}))$, which is absurd. It is also easy to see that for all $i_1, i_2 \in I$, with $i_1 \neq i_2$, $g_{i_1} g_{i_2} = 0$ and for each $j_1, j_2 \in J$, with $j_1 \neq j_2$, $f_{ij_1} f_{ij_2} = 0$. Consequently, the product of any two ideals in the above collection is zero which is sufficient to make the collection independent (note, $C(X)$ is reduced). Clearly, the cardinality of this collection is at least $\sum_{i\in I} |J_i|$, hence $\mathrm{Gdim}S_A^{-1}C(X) \geq \sum_{i\in I} \lambda_i = \lambda$ (note, $|J_i| = \lambda_i$ can occur). Finally we claim that the inequality, $\mathrm{Gdim}S_A^{-1}C(X) \geq \lambda$, gives us the equality. To see this, it is sufficient to prove that whenever $\{(\frac{f_t}{1}) : t \in T\}$ is an independent collection of nonzero ideals in $S_A^{-1}C(X)$ with $|T| > \lambda$, it leads us to a contradiction. Since $\frac{f_t}{1} \neq 0$, for all $t \in T$, we infer that $f_t \notin O_A$ for all $t \in T$, that is to say, $A \nsubseteq \mathrm{int}_X Z(f_t)$. Thus for each $t \in T$ there must exist $i \in I$ with $x_i \notin \mathrm{int}_X Z(f_t)$, hence $x_i \in \mathrm{cl}_X(X \setminus Z(f_t)$. Now for each $i \in I$, put $A_i = \{t \in T : x_i \in \mathrm{cl}_X(X \setminus Z(f_t))\}$ and it is evident that $T = \bigcup_{i\in I} A_i$. Since we have assumed that $|T| > \lambda$, there must exists $i \in I$

such that $|A_i| > \lambda_i$. In view of the fact that $\frac{f_{t_1}}{1}\frac{f_{t_2}}{1} = 0$ for $t_1, t_2 \in A_i$, $t_1 \neq t_2$, we infer that $f_{t_1}f_{t_2} \in O_A$, by Lemma 3.7, that is to say that $A \subseteq \mathrm{int}_X Z(f_{t_1}f_{t_2})$, hence $x_i \in \mathrm{int}_X Z(f_{t_1}f_{t_2})$, i.e., $f_{t_1}f_{t_2} \in O_{x_i} = O_{M_{x_i}}$. It is also manifest that $x_i \in \mathrm{cl}_X(X \setminus Z(f_t)$ for all $t \in A_i$. Consequently by the definition of the local cellularity at $x_i$, we have $c_l(x_i) \geq |A_i| > \lambda_i$. But $c_l(x_i) = \mathrm{Gdim}C(X)_{M_{x_i}} = \lambda_i$, by Theorem 3.4, which is the desired contradiction. □

The first part of Example 3.8, is now a trivial consequence of the following immediate corollary.

Corollary 3.10. *Let $A = \{x_1, x_2, \ldots, x_k\}$ be a subset of a space $X$ and each $x_i$ has a finite rank. Then $GdimS_A^{-1}C(X) = \sum_{n=1}^{k} GdimC(X)_{M_{x_n}} = \sum_{n=1}^{k} rk(x_n)$.*

The following examples show that for each cardinal number $\lambda$, there exist a topological space $X$ and a multiplicatively closed set $S$ with $\mathrm{Gdim}S^{-1}C(X) = \lambda$. The first example is the generalization of the fact that $\mathrm{Gdim}C(X) = c(X)$.

Example 3.11. Let $f \in C(X)$ and $S = \{f^n : n = 0, 1, \ldots\}$. We show that $\mathrm{Gdim}S^{-1}C(X) = c(X \setminus Z(f)) = \mathrm{Gdim}C(X \setminus Z(f))$. Suppose that $\{B_i : i \in I\}$ is an independent set of nonzero ideals of $S^{-1}C(X)$ and take $0 \neq \frac{f_i}{1} \in B_i$ for each $i \in I$. Hence $f_i f \neq 0$ which means that $(X \setminus Z(f)) \cap (X \setminus Z(f_i)) \neq \emptyset$. On the other hand, $\frac{f_i}{1}\frac{f_j}{1} = 0$ implies that $ff_if_j = 0$, $\forall i \neq j$, i.e., $[(X \setminus Z(f)) \cap (X \setminus Z(f_i))] \cap [(X \setminus Z(f)) \cap (X \setminus Z(f_j))] = \emptyset$. Therefore the set $\{(X \setminus Z(f)) \cap (X \setminus Z(f_i)) : i \in I\}$ is a collection of mutually disjoint nonempty open subsets of $X \setminus Z(f)$ which means that $\mathrm{Gdim}S^{-1}C(X) \leq c(X \setminus Z(f))$. Now let $\{G_i : i \in I\}$ be a collection of pairwise disjoint nonempty open sets in $X \setminus Z(f)$. Since $G_i$'s are also open in $X$, we may define $f_i \in C(X)$ such that $f_i(X \setminus G_i) = \{0\}$ and $f_i(x_i) = 1$ for some $x_i \in G_i$. Since $x_i \in X \setminus Z(f)$, clearly $f_i f \neq 0$, so $\frac{f_i}{1} \neq 0$, $\forall i \in I$. On the other hand, $f_i f_j = 0$ implies that $\frac{f_i}{1}\frac{f_j}{1} = 0$, $\forall i \neq j$. Therefore the collection $\{(\frac{f_i}{1}) : i \in I\}$ is an independent set of nonzero ideals in $S^{-1}C(X)$. This means that $\mathrm{Gdim}S^{-1}C(X) \geq c(X \setminus Z(f))$, hence $\mathrm{Gdim}S^{-1}C(X) = c(X \setminus Z(f))$. The second equality $c(X \setminus Z(f)) = \mathrm{Gdim}C(X \setminus Z(f))$ is proved in [1].

Example 3.12. Let $G$ be an open subset of $X$ and $T = C(X) \setminus \cup_{x \in G} M_x$. Then $\mathrm{Gdim}T^{-1}C(X) = c(G) = \mathrm{Gdim}C(G)$. Clearly $T = \{g \in C(X) : G \cap Z(g) = \emptyset\}$ and $\frac{f}{1} = 0$ if and only if $f \in O_G$. Since $G$ is open, clearly $O_G = M_G$. Suppose that $\{B_i : i \in I\}$ is an independent set of nonzero ideals in $T^{-1}C(X)$ and take $0 \neq \frac{f_i}{1} \in B_i$, $\forall i \in I$. Hence $f_i \notin O_G = M_G$ which means that $G \cap (X \setminus Z(f_i)) \neq \emptyset$, $\forall i \in I$. On the other hand $\frac{f_i}{1}\frac{f_j}{1} = 0$ implies that $f_i f_j \in O_G = M_G$, $\forall i \neq j$. Thus $G \subseteq Z(f_i) \cup Z(f_j)$ and in the other word, $[G \cap (X \setminus Z(f_i))] \cap [G \cap (X \setminus Z(f_j))] = \emptyset$, $\forall i \in I$. Therefore the collection $\{G \cap (X \setminus Z(f_i)) : i \in I\}$ is a collection of mutually disjoint nonempty open subsets of $G$. This means that $c(G) \geq \mathrm{Gdim}T^{-1}C(X)$. Now suppose that $\{H_i : i \in I\}$ is a collection of disjoint nonempty open subsets of $G$. For each $i \in I$, we define $f_i \in C(X)$ such that $f_i(X \setminus H_i) = \{0\}$ and $f_i(x_i) = 1$ for some $x_i \in H_i$ (note that $H_i$ is also open in $X$). Clearly $f_i \notin O_G = M_G$, i.e.,

$\frac{f_i}{1} \neq 0$, $\forall i \in I$. On the other hand $f_i f_j = 0$ implies that $\frac{f_i}{1}\frac{f_j}{1} = 0$ and hence the collection $\{(\frac{f_i}{1}) : i \in I\}$ is an independent set of nonzero ideals of $T^{-1}C(X)$. This shows that $\mathrm{Gdim}T^{-1}C(X) \geq c(G)$, so $\mathrm{Gdim}T^{-1}C(X) = c(G)$. The second equality $c(G) = \mathrm{Gdim}C(G)$ is proved in [1].

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