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Pure ideals, z-ideals and ideals with Artin-Rees property in $C(X)$

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Abstract. An ideal I in a commutative ring R is said to be pure if for each $a \in I$ there exists $b \in I$ such that $a = ab$ or equivalently, for each ideal J in R, the equality $I \cap J = IJ$ holds. I is called a z-ideal if for each $a \in I$, we have $M_a \subseteq I$, where M_a is the intersection of all maximal ideals in R containing a. Whenever for every ideal J in R, there exists $n \in \mathbb{N}$ such that $J^n \cap I \subseteq JI$, then we say that I has Artin-Rees property. It is clear that every pure ideal in $C(X)$ has Artin-Rees propertyis. Pure ideals are also z-ideals and a z-ideal in $C(X)$ is pure if and only if it has Artin-Rees property. In this note, we characterize spaces X for which every z-ideal of $C(X)$ has Artin-Rees property. We also observe that every z-ideal of $C(X)$ is pure if and only if X is a P-space (a space in which every G_δ-set is open). Regarding these ideals, some questions are given.

Introduction. We denote by $C(X)$ ($C^*(X)$) the ring of (bounded) real-valued, continuous functions on a completely regular Hausdorff space X. Whenever $C(X) = C^*(X)$, then X is called *pseudocompact*. βX is the *Stone-Čech compactification* of X and for any $p \in \beta X$, the maximal ideal M^p (resp., the ideal O^p) is the set of all $f \in C(X)$ for which $p \in \mathrm{cl}_{\beta X} Z(f)$ (resp., $p \in \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z(f)$). More generally, for $A \subseteq \beta X$, M^A (resp. O^A) is the intersection of all M^p (resp. O^p) with $p \in A$. For each $f \in C(X)$, the *zero-set* $Z(f)$ is the set of zeros of f and $X \setminus Z(f)$ is the *cozero-set* of f. An ideal I in $C(X)$ ($C^*(X)$) is called *fixed* if $\bigcap Z[I] = \bigcap_{f\in I} Z(f) \neq \emptyset$, else is *free*. The set of all fixed maximal ideals of $C(X)$ is exactly the set $\{M_x : x \in X\}$, where $M_x = \{f \in C(X) : f(x) = 0\}$. It is easy to see that for each $f \in C(X)$, $M_f = \{g \in C(X) : Z(f) \subseteq Z(g)\}$. This implies that an ideal I in $C(X)$ is a z-ideal if and only if $f \in I$, $g \in C(X)$ and $Z(f) \subseteq Z(g)$ imply that $g \in I$. Using this topological definition, M^A and O^A, for each subset A of βX are z-ideals and in particular, every maximal ideal and each O^p for $p \in \beta X$ is a z-ideal. Since every intersection of z-ideals is a z-ideal, for each $f \in C(X)$, M_f is also a z-ideal. Moreover if I is a z-ideal in $C(X)$ and $f^n \in I$ for some

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$n \in \mathbb{N}$, then $Z(f^n) = Z(f)$ implies that $f \in I$, i.e., every z-ideal in $C(X)$ is semiprime. It is well known that every pure ideal in $C(X)$ is of the form O^A for some $A \subseteq \beta X$, see [5], [6] and [7], so every pure ideal is a z-ideal. Now it is natural to ask when is every z-ideal in $C(X)$ a pure ideal? By definition, it is clear that every pure ideal has Artin-Rees property, but what about the converse? We answer these questions in the next section. Note that if I is a z-ideal, then $I^n = I$, for each $n \in \mathbb{N}$, hence every z-ideal with Artin-rees property is pure.

Results. Whenever I and J are z-ideals in $C(X)$, then $I \cap J = IJ$. In fact if $f \in I \cap J$, then $Z(f^{\frac{1}{3}}) = Z(f^{\frac{2}{3}}) = Z(f)$ implies that $f^{\frac{1}{3}} \in I$ and $f^{\frac{2}{3}} \in J$ for, I and J are z-ideals. Hence $f = f^{\frac{1}{3}} f^{\frac{2}{3}} \in IJ$. This means that whenever every ideal of $C(X)$ is a z-ideal, then every ideal of $C(X)$ is pure. the following Proposition states that if every z-ideal of $C(X)$ is pure, then all ideals of $C(X)$ are pure. we recall hat a space X is a P-space if every G_δ-set or equivalently, every zero-set in X is open. It is well known that X is a P-space if and only if every ideal of $C(X)$ is a z-ideal, see 4J in [9].

Proposition 1. The following statements are equivalent.

1. Every z-ideal of $C(X)$ is pure.
2. X is a P-space.
3. Every ideal of $C(X)$ is pure.

Proof. If every z-ideal of $C(X)$ is pure, then for each $f \in C(X)$, the z-ideal M_f is a pure ideal. Since $f \in M_f$, there exists $g \in M_f$ such that $f = gf$ or $f(1-g) = 0$. Since $g \in M_f$, we have $Z(f) \subseteq Z(g)$ and this means that if $x \in X$ and $f(x) = 0$, then $(1-g)(x) \neq 0$, i.e., $Z(f) \cap Z(1-g) = \emptyset$. On the other hand $f(1-g) = 0$ implies that $Z(f) \cup Z(1-g) = X$. Therefore $Z(f) = X \setminus Z(1-g)$, i.e., $Z(f)$ is open, so X is a P-space. Whenever X is a P-space, then every ideal of $C(X)$ is a z-ideal. But we have already observed, by the argument preceding the proposition, that every ideal of $C(X)$ is pure. Finally, if every ideal of $C(X)$ is pure, then clearly every z-ideal of $C(X)$ is pure and we are done.

As we already mensioned, every z-ideal with Artin-Rees property is pure. The following corollary is now an immediate consequence of this latter fact and Proposition 1.

Corollary 2. Every z-ideal of $C(X)$ has Artin-Rees property if and only if X is a P-space.

By 4J in [9], every ideal in $C(X)$ is a z-ideal if and only if X is a P-space and using Proposition 1, every ideal in $C(X)$ is pure if and only if X is a P-space. By the following corollary, we have the same topological characterization, in case every ideal in $C(X)$ has Artin-Rees property.

Corollary 3. Every ideal of $C(X)$ has Artin-Rees property if and only if X is a P-space.
Proof. If every ideal of $C(X)$ has Artin-Rees property, then every z-ideal of $C(X)$ has also this property and by Corollary 1, X is a P-space. Whenever X is a P-space, then by Proposition 1, every ideal of $C(X)$ is pure and hence every ideal of $C(X)$ has Artin-Rees property.

Concerning the ideals of the title, it remains the following questions:

Questions.
1. When is every ideal in $C(X)$ with Artin-Rees property a z-ideal?
2. When is every ideal in $C(X)$ with Artin-Rees property a pure ideal?

An ideal I in a ring R is said to be z°-ideal if for each $a \in I$, we have $P_a \subseteq I$, where P_a is the intersection of all minimal prime ideals of R containing a. It is well known that for each $f \in C(X)$, $P_f = \{g \in C(X) : \text{int}_X Z(f) \subseteq \text{int}_X Z(g)\}$. Using the definition of a z°-ideal, it is clear that every element of a z°-ideal is a zero divisor. Every z°-ideal in $C(X)$ is a z-ideal but not conversely. For more information about the z°-ideals in reduced rings and $C(X)$, see [1], [2], [3], [4] and [8]. Now the following question is also natural.

Question. When is every ideal in $C(X)$ with Artin-Rees property a z°-ideal?

It is known that the sum of z-ideals in $C(X)$ is a z-ideal and the sum pure ideals is also pure, see [9], [10] and [7]. In the following result, we give a proof for the purity of the sum of two pure ideals.

Proposition 1. The sum of every two pure ideals is pure.
Proof. Let I and J be two pure ideals. Suppose that $f \in I+J$, hence $f = i+j$ for some $i \in I$ and $j \in J$. Since I and J are pure, $i = ig$ and $j = jh$ for some $g \in I$ and $h \in J$. Now $f = i+j = gi+hj$ implies that $fgh = ih+jg$ and hence $f(g+h) = (i+j)(g+h) = i+j+ih+gj = f+fgh$. So $f = f(g+h-gh)$, where $g+h-gh \in I+J$, i.e., $I+J$ is pure.

We conclude this article by the following question.

Question. Let I and J be two ideals in $C(X)$ with Artin-Rees property. Is $I+J$ an ideal with Artin-Rees property?

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